Topological Poincaré conjecture in dimension 4 (the work of M. H. Freedman)

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French to English translation by Min Hoon Kim and Mark Powell.

Introduction

M. Freedman proved that every smooth homotopy 4-sphere M^4 is homeomorphic to S^4. Our main goal is to give an exposition of his proof. (In this paper, every manifold will be metrisable and finite dimensional.) We do not know yet whether such an M^4 is always diffeomorphic to S^4. On the other hand, Freedman proved that every *topological* homotopy 4-sphere M^4 (without any given smooth structure) is actually homeomorphic to S^4 (see below).

H. Poincaré made a conjecture according to which every smooth, homotopy n-sphere M^n is diffeomorphic to S^n. The first non-trivial case, of dimension 3, remains open despite of ceaseless efforts of innumerable mathematicians. An amusing detail is the counterexample of J. H. C. Whitehead [Whi35b]; his own *false* proof of this conjecture plays large role in this lecture (see Section 2).

J. Milnor discovered [Mil56] smooth manifolds M^7 which are homeomorphic to S^7 but not diffeomorphic to S^7 (such *exotic spheres* exist in dimension ≥ 7 [KM63]). Therefore the above Poincaré conjecture has to be revised for dimension ≥ 7. S. Smale [Sma61] established his theory of handles to prove that every smooth homotopy n-sphere is homeomorphic to S^n for $n \geq 6$. His technical result, the h-cobordism theorem (see below) is more precise. By combining this with surgery techniques of Kervaire-Milnor [KM63] establishes $n = 6$ and 5 cases of the above Poincaré conjecture. M. Newman adapted the *engulfing* method of J. Stallings to prove the purely topological version, that is, every topological homotopy n-sphere is homeomorphic to S^n if $n \geq 5$. (Smale's surgery method has also been adapted to the topological category [KS77].) In summary, Poincaré conjecture is essentially resolved in dimension ≥ 5, is not resolved in dimension 3 and is partially resolved in dimension 4.

We sketch a proof of Freedman's theorem which implies the topological classification of smooth, simply-connected closed 4-manifolds and many other results of the fundamental importance. Let V and V' be two such manifolds. Suppose that there is an isomorphism $\Theta\colon H_2(V) \to H_2(V')$ which preserves the intersection forms. (Note that V is a homotopy 4-sphere if and only if $H_2(V) = 0$.)

Theorem A. *In this situation, Θ is realised by a homeomorphism $V \to V'$.*

Proof. It is not difficult to realise Θ by a homotopy equivalence $g\colon V \to V'$ [MH73]. Surgery theory [Wal64, Bro72] gives a compact 5-manifold W with boundary $\partial W = V \sqcup -V'$ such that the inclusions $V \to W$ and $V' \to W$ are homotopy equivalences and such that the restriction $r|_V\colon V \to V'$ of the retraction $r\colon W \to V'$ is homotopic to g. The compact triad $(W; V, V')$ is called an *h-cobordism.* Smale's theory of handles tries to improve a Morse function $f\colon (W; V, V') \to ([0,1]; 0, 1)$ to obtain a situation where f has no critical points, that is, f is a smooth submersion. Then W is a fibre bundle over $[0,1]$ (a remark of Ehresmann) and hence W is diffeomorphic to $V \times [0,1]$. We are going to find a topological submersion f which shows that W is a topological fibration on I (see [Sie72b, Section 6]) so that W is homeomorphic to $V \times [0,1]$. □

In particular, we will prove the simply connected, topological 5-dimensional h-cobordism theorem.

Theorem B. *Every smooth, compact, simply connected, 5-dimensional h-cobordism $(W; V, V')$ is topologically trivial. That is, W is homeomorphic to $V \times [0,1]$.*

For $n \geq 6$, instead of 5, Smale's h-cobordism theorem gives the stronger conclusion that W is diffeomorphic to $V \times [0,1]$. In dimension 5, his methods apply, but leaving to prove that W is diffeomorphic to $V \times [0,1]$. The following problem is not yet resolved:

Remaining smooth problem. Let $S = S_1 \sqcup \cdots \sqcup S_k$ and $S' = S'_1 \sqcup \cdots \sqcup S'_k$ be 2 families of disjointly embedded 2-spheres in a simply connected 4-manifold M (in fact f^{-1}(a point)) in such a way that the homological intersection number $S_i \cdot S'_j = \pm\delta_{i,j}$. Can one reduce $S \cap S'$ to k points of intersection (smooth and transverse) by a smooth isotopy of S in M?

Similarly, to obtain the fact that W is homeomorphic to $V \times [0,1]$, we claim (see [Mil65] and [KS77, Essay III]) that it suffices to solve the following problem:

Remaining topological problem (resolved here)**.** With the data of the smooth problem, reduce $S \cap S'$ to k points by a topological isotopy of S in M, that is given by an ambient isotopy h_t, $1 \leq t \leq 1$, of $\mathrm{id}|_M$ fixing a neighbourhood of k-points of $S \cap S'$.

Whitney introduced a natural method for solving these problems. In the model $(\mathbb{R}^2; A, A')$, (this is a straight line A cutting a parabola A' in 2-points), we can disengage A from A' by a smooth isotopy *with compact support* (that is, fixing a neighbourhood of ∞). One eliminates thus the 2 intersection points. We deduce that in the stabilised Whitney model,

$$(\mathbb{R}^4; A_+, A'_+) = (\mathbb{R}^2 \times \mathbb{R}^2; A \times 0 \times \mathbb{R}, A' \times \mathbb{R} \times 0),$$

there is an isotopy with compact support that makes the plane A_+ disjoint from the plane A'_+, deleting the 2 transverse intersection points between A_+ and A'_+.

We call a smooth (resp. topological) *Whitney process*, a smooth embedding (resp. a topological embedding) of a disjoint union of copies of the model $(\mathbb{R}^4; A_+, A'_+)$, whose image contains $S \cap S' -$ (k points). Such a procedure would clearly give the demanded isotopy to resolve the remaining smooth problem (respectively, the remaining topological problem).

Theorem C (Casson-Freedman)**.** *In this context, after a preliminary smooth isotopy of S in M, (adding intersection points with S' by finger moves, far from $S \cap S'$), the topological Whitney process becomes possible.*

The first step of the proof (1973–1976) is due to A. Casson. Let B be a smooth, compact 2-disc in the boundary component of $\mathbb{R}^2 - A \cup A'$. The product $B \times \mathbb{R}^2$ is an open, embedded 2-handle (as a closed submanifold) in the Whitney model, and disjoint from $A_+ \cup A'_+$. In $B \times \mathbb{R}^2$, Casson constructed certain open sets $H = B \times \mathbb{R}^2 - \Omega$ with boundary $\partial H = \partial B \times \mathbb{R}^2$, that we call *open Casson handles.* (See Section 2 for the precise definition). We are again unable to decide whether H is diffeomorphic to $B \times \mathbb{R}^2$ or not. Replacing $B \times \mathbb{R}^2$ by $H \subset B \times \mathbb{R}^2$ in this Whitney model $(\mathbb{R}^4; A_+, A'_+)$ we have an open set $(\mathbb{R}^4 - \Omega; A_+, A'_+)$, that we call the Whitney-Casson model. By a remarkable infinite process, Casson proved:

Theorem D (Casson [Cas86], compare [Man80])**.** *After a preliminary smooth isotopy of S in M, one can find in $(M; S, S')$ smoothly embedded, disjoint Whitney-Casson models so that the models contain all the points of $S \cap S'$ except the k intersection points.*

The theorem of Casson and Freedman now follows from the theorem that we will discuss.

Theorem E (Freedman, 1981)**.** *Every open Casson handle is homeomorphic to $B^2 \times \mathbb{R}^2$. Therefore, the Whitney model $(\mathbb{R}^4; A_+, A'_+)$ is homeomorphic to $(\mathbb{R}^4 - \Omega; A_+, A'_+)$.*

The non-compact version of Theorem B is also important.

Theorem F. *Let $(W; V, V')$ be a simply connected, proper smooth 5-dimensional h-cobordism with a finite number of ends and a trivial π_1-system at each end. Then W is homeomorphic to $V \times [0,1]$.*

The difficult proof proposed by Freedman (October 1981) initiates the proof of proper s-cobordism theorem sketched in [Sie70], while avoiding to do 2 Whitney processes, in view of the loss of differentiability occasioned by Theorem C.

This gives (compare [Fre79] and [Sie80]) the topological classification of closed, simply connected topological 4-manifolds that admit (do they all?) a smooth structure in the complement of a point. They are classified by their intersection form on H_2, together with the Kirby-Siebenmann obstruction x [KS77]; every unimodular forms over $\mathbb{Z}$ is realised, as well as every $x \in \mathbb{Z}_2$, except that for even forms, $x \in \sigma/8 \in \mathbb{Z}_2$. Every topological 4-manifold V which is homotopy equivalent to S^4 is in this class, because $V -$ (point) is contractible and thus $V -$ (point) can be immersed into $\mathbb{R}^4$ (compare [KS77]).

It also follows (see [Fre79, Sie80]) that every smooth homology 3-sphere V (that is, $H_*(V) \cong H_*(S^3)$) is the boundary of a contractible topological 4-manifold W.

Report

Mike Freedman announced his proof of the topological Poincaré conjecture in August 1981 at the AMS conference at UCSB where D. Sullivan was giving a lecture series on Thurston's hyperbolization theorem. His argument was very brilliant, but nevertheless watertight.

A large group of conversations then formulated certain objections, which proposed up it the statement of Approximation theorem 5.1. However, Freedman already had in his head his trick of replication, and in some changes, his imposing formal proof was born.

In the meantime, R. D. Edwards had found a mistake in the shrinking arguments (see Section 4) who is a great expert in this method. He repaired this mistake before it was announced. (I think that he introduced in particular the relative shrinking arguments.) At the end of October 1981, Freedman explained the details of his proof, with charm and patience, at a special conference at University of Texas at Austin (the school of R. L. Moore) before an audience of specialists, including, in the place of honour, Casson and R. H. Bing, creators of the two theories essential in the proof.

This paper relates the proof given in Texas, with improvements in detail added in backstage. Already in 1981, R. Ancel [Anc81] has clarified and improved the complexities in bookkeeping of the approximation theorem 5.1. In particular, he was able to reduce a hypothesis of Freedman demanding that the pre-images of the singular point constitute a null decomposition, showing that $S(f)$ countable and of *dimension* 0 [Edw75] suffices. J. Walsh contributed certain simplifications to the shrinking arguments (end of Section 4). W. Eaton suggested to me the 4-balls that help to understand relative shrinking (Lemma 4.9 and Proposition 4.11). I proposed a global coordinate system of a Casson handle. (It was initially necessary to embed in there, the frontier of a handle.)

My exposition (January 1982) does not seem to have changed essentially from my memories of Texas. Only my construction of corrective 2-discs (the $D(\alpha)$ of Section 3.9) deviates, probably for reasons of taste. I am indebted to A. Marin for his brotherly and insightful comments.

1. Terminology

These terminologies are used from now on except when otherwise indicated. All spaces admit a metric, denoted generally by d. Maps are all continuous. The support of a map $f\colon X \to X$ is the closure of $\{x \in X \mid f(x) \neq x\}$. The support of a homotopy, or an isotopy $f_t\colon X \to X$ $(0 \leq t \leq 1)$ is the closure of

$$\{x \in X \mid f_t(x) \neq x \text{ for some } t \in [0,1]\}.$$

For a subset A, define the closure $\overline{A}$, the interior $\mathring{A}$ and the frontier δA, always with respect to the understood ambient space (the largest involved). If A is a manifold, it is often necessary to distinguish $\mathring{A}$ from its formal interior $\operatorname{int} A$ and δA from the formal boundary ∂A.

A decomposition $\mathcal{D}$ of a space X will be a collection of compact disjoint subsets in X that is usc (upper semi continuous); the quotient space $X/\mathcal{D}$ is obtained by identifying each element of $\mathcal{D}$ to a point (see [MV75] for a metric). The quotient map is $X \to X/\mathcal{D}$ is closed, which is exactly equivalent to the usc property.

The set of connected components of a space X is denoted by $\pi_0(A)$. If A is compact, $\pi_0(A)$ is at the same time a decomposition of A for which the quotient $A/\pi_0(A)$ is a compact set of dimension 0 (totally discontinuous), that is identified with $\pi_0(A)$ as a set. If $A \subset X$, $\pi_0(A)$ gives a decomposition of X whose quotient space is denoted by $X/\pi_0(A)$. The *endpoint compactification* will appear in Section 2.

The manifolds and submanifolds mentioned will be (unless otherwise indicated) smooth. For manifolds, we adopt the usual convention [KS77, Essay I]; in particular, $\mathbb{R}^n$ is the Euclidean space with the metric $d(x,y) = |x-y|$; $B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$. $I = [0,1]$. A *multi-disc* is a disjoint union of finitely many discs (each are diffeomorphic to B^2). Similarly, for multi-handle, etc. The symbols $\cong$, $\approx$ and $\simeq$ indicate a diffeomorphism, a homeomorphism and a homotopy equivalence, respectively.

2. Casson tower and Freedman's mitosis

We will use 2 versions B^2 and D^2 of the standard smooth 2-disc $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$. The standard 2-handle is $(B^2 \times D^2, \partial B^2 \times D^2)$; its *attaching region* ∂_- is $\partial B^2 \times D^2$; its *skin* ∂_+ is $B^2 \times \partial D^2$, its *core* is $B^2 \times 0$. A 2-*handle* is a pair $(H^4, \partial_- H)$ diffeomorphic to $(B^2 \times D^2, \partial B^2 \times D^2)$. An open 2-handle is a manifold diffeomorphic to $B^2 \times \mathring{D}^2$. For a 2-handle (possibly open), the *attaching region*, the *skin* and the *core* are defined by a diffeomorphism with the standard 2-handle (perhaps the open one). In this paper, we can allow ourselves to omit the prefix "2-"; handles of index $\neq 2$ appear rarely. Also, we write $\mathring{D}^2$ where we ought strictly to write int D^2.

A *defect* X in a handle $(H^4, \partial_- H)$ is a compact submanifold X of $H^4 - \partial_- H$ such that

(1) $(X, X \cap \partial_+ H)$ is a handle where $\partial_+ H$ is the skin of the handle $(H, \partial_- H)$.
(2) $(\partial_+ H, X \cap \partial_+ H)$ is (degree ± 1) diffeomorphic to the Whitehead double $(B^2 \times S^2, i(B^2 \times S^1))$ illustrated in Figure 1:
(3) In the 4-ball H^4 (with rounded corners), the core A^2 of the handle $(X, X \cap \partial_+ H)$ is an unknotted disc, that is, (H, A) is diffeomorphic to (B^4, B^2).

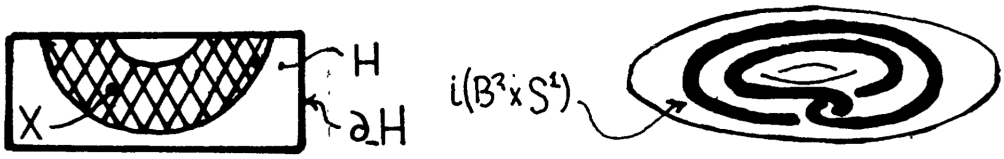


FIGURE 1.

A *multi-defect* X in a handle $(H^4, \partial_- H)$ is a finite sum and union $\sqcup_i X(i) = X$ of ≥ 1 defects $X(i)$ such that for an identification $(H^4, \partial_- H)$ with $(B^2 \times D^2, \partial B^2 \times D^2)$, project to B^2 the same number of disjoint discs in int B^2. A *multi-handle* $(H^4, \partial_- H^4)$ is a disjoint, finite sum of handles. A multiple defect $X \subset H^4$ in a multiple handle is a compact subset that gives rise, by intersection, to a multi-defect in each handle. With this data, we have:

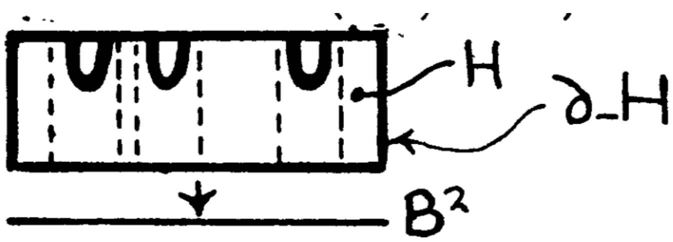


FIGURE 2.

Lemma 2.1. *The triad $(H^4 - \mathring{X}; \partial_- H, \delta X)$ determines H^4 and X in the following sense: if X' is a multi-defect in a handle $(H', \partial_- H')$ and $\theta\colon (H - \mathring{X}; \partial_- H, \delta X) \to (H' - \mathring{X}'; \partial_- H', \delta X')$ is a diffeomorphism, there exists a diffeomorphism $\Theta\colon H \to H'$ extending θ.*

Sketch of proof (*see* [Cas86]). If we attach a multi-handle $(X', \partial_- X')$ to $H - \mathring{X}$ along the frontier δX, in such a way that there exists no extension of θ to a diffeomorphism $\Theta\colon H \to (H - \mathring{X}) \cup X' = H'$, we claim that $(\partial H', \partial_- H)$ is diffeomorphic to $(S^3, \text{solid torus})$ where the solid torus is tied in a non-trivial knot. In fact, a connected sum of k non-trivial knots of the forms, $1 \leq k \leq |\pi_0(X)|$: $\square$

FIGURE 3. Twist knots

A *residual defect* Ω in a handle $(H^4, \partial_- H^4)$ is the intersection of a sequence

$$X_1 \supset \mathring{X}_1 \supset X_2 \supset \mathring{X}_2 \supset X_3 \supset \cdots$$

of compact submanifolds of $H^4 - \partial_- H^4$ such that, for all k, $(X_k, \delta X_k)$ is a multi-handle in which X_{k+1} is a multi-defect. The sequence $X_1 \supset X_2 \supset \cdots$ is called a russian doll of defects.

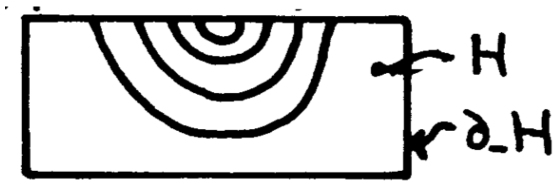


FIGURE 4.

A *Casson handle* is a pair $(H^4_\infty, \partial_- H^4_\infty)$ such that there exists a handle $(H, \partial_- H)$ with a residual defect $\Omega \subset H$ and an open smooth embedding $i_\infty\colon H_\infty \to H$ with image $H - \Omega$, which induces a diffeomorphism $i_\infty|\colon \partial_- H_\infty \to \partial_- H$. In other words, $(H_\infty, \partial_- H_\infty)$ is diffeomorphic to $(H - \Omega, \partial_- H)$.

The data of $(H, \partial_- H)$, the russian doll of defects X_i and $i_\infty\colon H_\infty \to H$, constitute what we will call a *presentation of a Casson handle* $(H_\infty, \partial_- H_\infty)$. We will also denote $H_k = i_\infty^{-1}(H - \mathring{X}_k)$ and $\partial_- H_k = \partial_- H_\infty$. Then, $H_\infty = \cup_k H_k$. The manifold H_k is called a tower of height k, its stages are $E_j = i_\infty^{-1}(X_{j-1} - X_j)$ for $j \leq k$. The restriction of i_∞ to H_k will be denoted $i_k\colon H_k \to H$.

The skin of $(H_\infty, \partial_- H_\infty)$ is $\partial_+ H_\infty = i_\infty^{-1}(\partial_+ H)$; moreover, by taking intersection with $\partial_+ H_\infty$, we define the skin $\partial_+ H_k$ of H_k and $\partial_+ E_k$ of E_k. Similarly $\partial_+ X_k = X_k \cap \partial_+ H$.

A Casson handle $(H_\infty, \partial_- H_\infty)$ is never compact; we will often encounter the endpoint compactification $\widehat{H}_\infty$ of H_∞. Recall that the endpoint compactification $\widehat{M}$ of a connected, locally connected and locally compact space M is the Freudenthal compactification that adds to M the compact 0-dimensional space $\mathrm{Ends}(M)$ which is the (projective) limit of an inverse system $\{\pi_0(M - K) \mid K \subset M \text{ such that } K \text{ is compact}\}$.

$\widehat{H}_\infty$ is identified (by i_∞) with the quotient of H^4 obtained by crushing each connected compact of Ω to a point. (To verify this, note that $\pi_0(\Omega)$ with the compact topology is the (projective) limit of an inverse system $\{\pi_0(U) \mid U \text{ is an open subset of } H \text{ containing } \Omega\}$.

We remark that $\widehat{H}_\infty$ is the Alexandroff compactification by a point, exactly when $\Omega \subset H$ is connected, or if each successive multiple defect X_i is a single defect. The reader who feels discombobulated by all the complexities to come may be interested in restricting themselves at first to this cases, which already contains all the geometric ideas.

$\widehat{H}_\infty$ has all the local homological properties of a manifold; it is what we call a homology manifold. But its formal boundary, the closure of ∂H_∞, is not a topological manifold near its ends. For example, if Ω is connected, by definition, ∂H_∞ (which is homeomorphic to $\partial H - \partial_+ \Omega$) is one of the contractible 3-manifolds of J. H. C. Whitehead [Whi35b, Whi35a], with a non-trivial π_1-system at infinity. $\partial_+\Omega \subset \partial H \cong S^3$ is a *Whitehead compactum.* In the general case, $\partial_+\Omega$ is called a *ramified Whitehead compactum.* Thus, $(\widehat{H - \Omega}, \partial_- H)$ has no chance of being a topological handle. On the other hand, $H - (\partial_+ H \cup \Omega)$ is homeomorphic to $B^2 \times \mathbb{R}^2$; this will be the central result of this paper.

Theorem 2.2 (Freedman 1981)**.** *Every open Casson handle M is homeomorphic to $B^2 \times \mathbb{R}^2$.*

The proof of Theorem 2.2 starts with a result of 1978, when Freedman was able to construct a smooth 4-manifold M without boundary which is not homeomorphic to $S^3 \times \mathbb{R}$ that is however the image of a proper map of degree ± 1, $S^3 \times \mathbb{R} \to M$ (see [Fre79] and [Sie80]).

A Casson tower of height k, or more briefly C_k, is a pair diffeomorphic to $(H - \mathring{X}_k, \partial_- H)$ where $X_1 \supset X_2 \supset \cdots$ is a russian doll of defects in a handle $(H, \partial_- H)$.

Theorem 2.3 (Mitosis (a finite version))**.** *Let $(H_6, \partial_- H_6)$ be a Casson tower C_6 of height 6. There is a C_{12}, or $(H'_{12}, \partial_- H'_{12})$, such that*

1. *$\partial_- H'_{12} = \partial_- H_6$.*
2. *$H'_{12} - \partial_- H_6 \subset \operatorname{int} H_6$.*
3. *$H'_{12} - H'_6$ is contained in a disjoint union of balls in $\operatorname{int} H_6$, one ball for each connected component.*

Condition ((3)) is related to the fact that, for each Casson tower $(H_k, \partial_- H_k)$, the manifold H_k can be expressed as a regular neighbourhood of a 1-complex, compare [Cas86]. Here is a schematic diagram of Freedman which summarises Theorem 2.3.

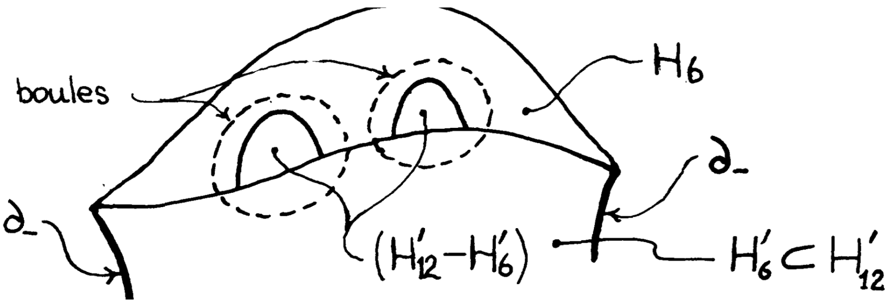


Figure 5.

In Section 3, Figure 6 will represent a C_6, and Figure 7 will represent a C_{12}, etc. From the point of view of the representation of corners on the boundary, it might be better to use Figure 8.

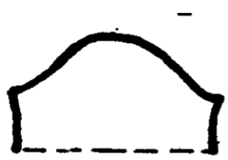


Figure 6.

The method of Freedman [Fre79] (compare [Sie80]) allows one to give a proof of Theorem 2.3. However, it is slightly more detailed than the analogues in [Fre79], [Sie80]. We will not cover this point in this paper (see [GS84] for an excellent write up of the Mitosis theorem 2.3).

Remark. Every pair $(k, 2k)$, $k > 6$, in place of $(6, 12)$ gives a statement that one can deduce without too much pain and sorrow that we could use in place of Theorem 2.3 in what follows.

Since we are going to use Theorem 2.3 often, it is convenient to make the following:

Figure 7.

Figure 8.

Change of notation 2.4. From now on, we write H_k and X_k in place of H_{6k+6} and X_{6k+6}, $k = 0, 1, 2, \ldots$. (Also the meaning of $E_k = H_k - H_{k-1}$, i_k, etc is changed.)

Theorem 2.5 (Mitosis (an infinite version))**.** *Let $(H_\infty, \partial_- H_\infty)$ be a Casson handle presented as above, and let $k \geq 0$ be an integer. There exists another Casson handle $(H'_\infty, \partial_- H_\infty) \subset (H_\infty, \partial_- H_\infty)$ satisfying the conditions.*

(1) $H'_{k-1} = H_{k-1}$ *if* $k \geq 1$.
(2) $\overline{H}'_\infty - H'_{k-1} \subset (\operatorname{int} H_k) - H_{k-1}$.
(3) *The closure* $\overline{H}'_\infty$ *of* H'_∞ *in* H_∞ *is the endpoint compactification of* H'_∞.

This infinite version, Theorem 2.5, follows from the finite version, Theorem 2.3, by an infinite repetition. One sufficiently shrinks balls given by Theorem 2.3 to ensure the condition ((3)) of Theorem 2.5.

3. Architecture of topological coordinates

The ambitious construction to come applies the mitosis theorem 2.5 and elementary geometry, to convert Theorem 2.2, that every open Casson handle is homeomorphic to $B^2 \times \mathbb{R}^2$, to two theorems on approximation by homeomorphisms. For Casson handles, we will use the terminology of Section 2, under the modified form in Change of notation 2.4 (by a reindexing).

The open Casson handle M will be identified to $N - \partial_+ N$ where $(N, \partial_- N)$ is a Casson handle (not open). Let $\widehat{N}$ be the endpoint compactification of N. Subtracting from N the (topological) interior of a collar neighbourhood of $\partial_+ N$ in N, very pinched towards the ends of N, we obtain a Casson handle $(H_\infty, \partial_- H_\infty) \subset (M, \partial M) \subset (N, \partial_- N)$ whose closure in $\widehat{N}$ is the endpoint compactification $\widehat{H}_\infty$ of H_∞. We fix a presentation of $(H_\infty, \partial_- H_\infty)$.

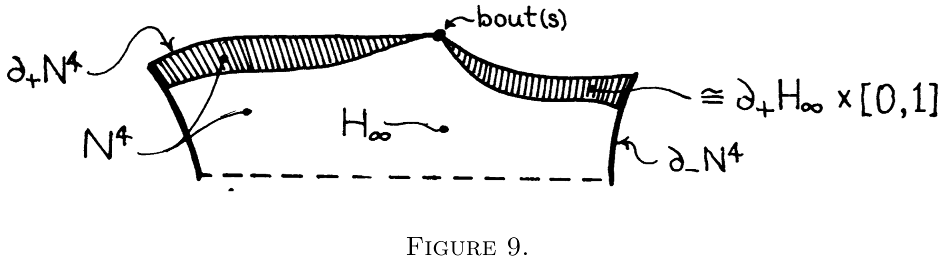


Figure 9.

We will construct a ramified system of Casson handles in $(N, \partial_- N)$, that, in some way, explores its interior.

3.1. Construction

For each finite sequence $(a_1, \dots, a_k)$ in $\{0, 1\}$ (finite dyadic sequence), we can define a presented Casson handle $(H_\infty(a_1, \dots, a_k), \partial_- H_\infty)$ contained in $(H_\infty, \partial_- H_\infty)$, whose presentation consists of an embedding $i_\infty(a_1, \dots, a_k)\colon H_\infty(a_1, \dots, a_k) \to B^2 \times D^2$, and a russian doll of defects $X_i(a_1, \dots, a_k)$, in the standard handle $B^2 \times D^2$ such that (for (1)–(5), see the right figure of Figure 10)

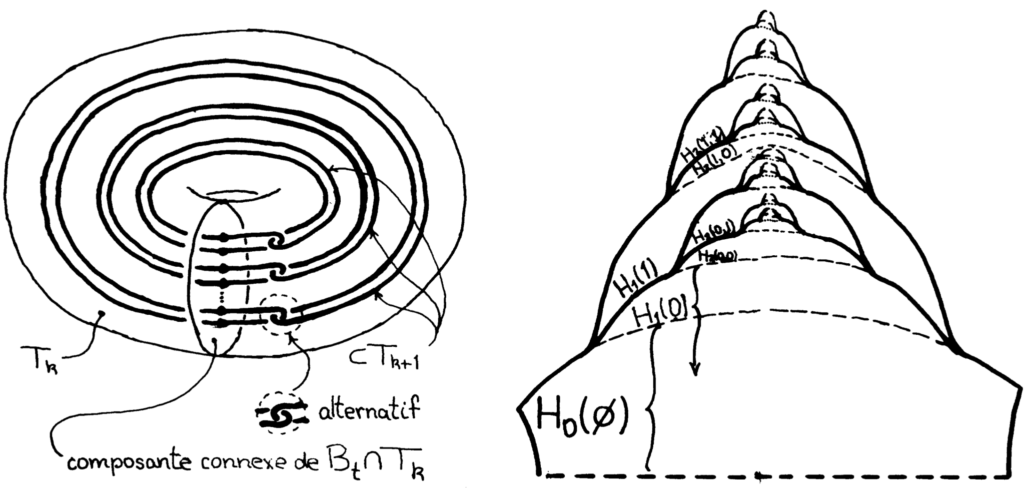


Figure 10.

(1) $H_\infty = H_\infty(\emptyset)$ (case $k = 0$) as a presented Casson handle.
(2) $H_\infty(a_1, \dots, a_k, 1) = H_\infty(a_1, \dots, a_k)$.
(3) $H_k(a_1, \dots, a_k, 0) = H_k(a_1, \dots, a_k)$ (recall that H_k are sets of 6-stages).
(4) The closure $\overline{H}_\infty(a_1, \dots, a_k, 0)$ in $\widehat{H}_\infty$ is an endpoint compactification of $H_\infty(a_1, \dots, a_k, 0)$.
(5) $\overline{H}_\infty(a_1, \dots, a_k, 0) - H_k(a_1, \dots, a_k) \subset \mathring{H}_{k+1}(a_1, \dots, a_k) - H_k(a_1, \dots, a_k)$.
(6) $i_k(a_1, \dots, a_k, 0) = i_k(a_1, \dots, a_k)$, so $X_k(a_1, \dots, a_k, 0) = X_k(a_1, \dots, a_k)$.
(7) The intersection of $X_{k+1/6}(a_1, \dots, a_k, 0)$ and $X_{k+1/6}(a_1, \dots, a_k)$ are empty, and their union is a multiple defect in $X_k(a_1, \dots, a_k)$. We also require a coherence condition on the total russian doll assumed by (7), that is to say $\{X_k\}$, where $X_k = \cup X_k(a_1, \dots, a_k)$. To formulate it, we momentarily suspend the reindexing convention 2.4 and into $T_k = \partial_+ X_k$.
(8) (without change of notation 2.4) There exists an interval $J \subset \partial D^2$ such that, for all $t \in J$, the meridional disc $B_t = B^2 \times t$ of the solid torus $B^2 \times \partial D$ meets the multiple solid tori T_k ideally, in the sense that each connected component of $B_t \cap T_k$ is a meridional disc of T_k, that meets T_{k+1} in an ideal fashion illustrated in the left figure of Figure 10:

Execution of Construction 3.1 (*by induction on k*). We start with $H_\infty(\emptyset) = H_\infty$. Having defined presented handle for every sequence of length $\leq k$ ($k \geq 6$), we define them for every sequence $(a_1, \dots, a_k, 1)$ by (2). Next, we define $H_\infty(a_1, \dots, a_k, 0)$ by Mitosis theorem 2.5 (infinite version). This assures that conditions (3), (4) and (5) are met. It remains to define the presentation of the Casson handle $(H_\infty(a_1, \dots, a_n, 0), \partial_- H_\infty)$ in such a fashion that the two last conditions (6) and (7) are satisfied. To define $i_\infty(a_1, \dots, a_k, 0)$, it is convenient to graft, onto $i_k(a_1, \dots, a_k)$, a presentation the near part of the Casson handle $(H_\infty(a_1, \dots, a_k, 0), \partial_- H_\infty)$, to know the Casson multihandle $(H_\infty(a_1, \dots, a_k, 0) - \mathring{H}_k(a_1, \dots, a_k, 0), \delta H_k(a_1, \dots, a_k, 0))$, where exceptionally $^\circ$ and δ denote the interior and the frontier in $H_\infty(a_1, \dots, a_k, 0)$ rather than in $\widehat{N}$. The graft is done with the help of Lemma 2.1. The last condition (7) is assured afterwards by an isotopy in $\mathring{X}_k(a_1, \dots, a_k)$. Having (1) to (7), the reader will know how to arrange that (8) is also satisfied. $\square$

Remark. If $(a_1, a_2, \dots)$ is an infinite sequence in $\{0, 1\}$, the union

$$H_\infty(a_1, a_2, \dots) = \cup_k H_\infty(a_1, a_2, \dots, a_k)$$

gives a Casson handle with an obvious presentation; moreover, the closure $\overline{H}_\infty(a_1, a_2, \dots)$ is the endpoint compactification (exercise). Thus, we have a vast collection of Casson handles in N, conveniently nested.

Of the system of handles $(H_\infty(a_1, \dots, a_k), \partial_- H_\infty)$, we especially use their skins $\partial_+ H_\infty(a_1, \dots, a_k)$. The union $P^3 = \cup \partial_+ H_\infty(a_1, \dots, a_k)$ of all its skin is what one calls a *branched manifold* in N^4, since near every point $P^3 - \partial_- H_\infty$, the pair (N^4, P^3) is C^1-isomorphic (same as C^∞-isomorphic, after some work that we leave to the reader) to the product of $\mathbb{R}^2$ with the model of branching $(\mathbb{R}^2, Y^1)$:

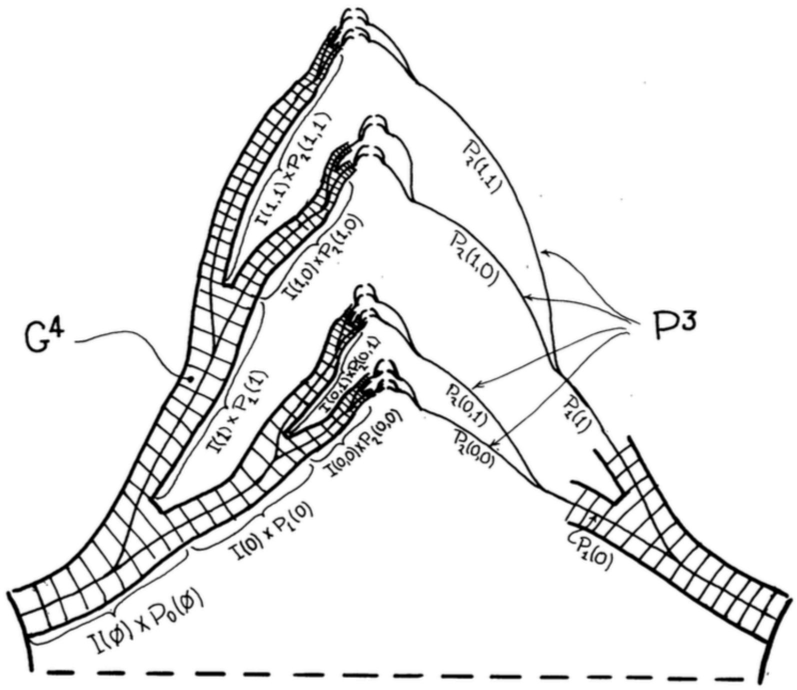


Figure 11.

where Y^1 is the union of two smooth curves (isomorphic to $\mathbb{R}^1$), properly embeded in $\mathbb{R}^2$ and which have in common exactly one closed half-line. One observes without difficulty that the closure $\overline{P}$ of P in $\widehat{N}$ is the endpoint compactification of P.

The branched manifold P splits along the singular points into compact manifolds:

$$P_k(a_1, \dots, a_k) = \partial_+ E_k(a_1, \dots, a_k) = E_k(a_1, \dots, a_k) \cap \partial_+ H_\infty(a_1, \dots, a_k).$$

Thus, $P_k(a_1, \dots, a_k)$ is the skin of the kth stage of $(H_\infty(a_1, \dots, a_k), \partial_- H_\infty)$.

3.2. Construction of the design G^4 (see Figure 11)

For P^3, we construct a neighbourhood G^4 in N^4 called the *design*, which has a decomposition $\mathcal{I}$ of G^4 into disjoint interval, of the sort that:

(1) For every interval I_α of $\mathcal{I}$, the intersection $I_\alpha \cap \partial_- N$ is I_α or the emptyset. A neighbourhood of I_α in $(G^4, P^3; \mathcal{I})$ is isomorphic to the product of $\mathbb{R}^2$ with an open 2-dimensional model $(G^2, P^1; \mathcal{I}')$ as follows:

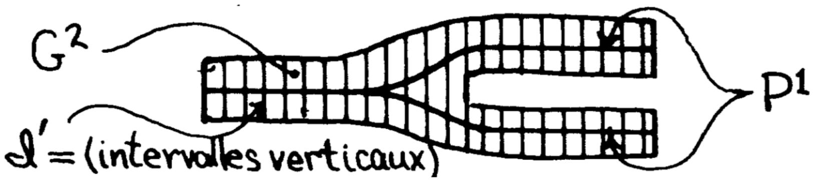


FIGURE 12.

(2) The closure $\overline{G}$ of G in $\widehat{N}$ is its endpoint compactification, and hence coincides with $G \cup \overline{P}$. It follows by combining, quite naively, two bicollars of genuine submanifolds of P^3. On the other hand, we clearly are permitted to suppose that G^4 contains the collar $N - \mathring{H}_\infty$ of $\partial_+ N$.

The design $(G^4, \mathcal{I})$ decomposed in intervals in a canonical fashion (along the 3-manifold formed by the exceptional intervals of $\mathcal{I}$ having interior points on ∂G^4) into genuine trivial I-bundles $I(a_1, \ldots, a_k) \times P_k(a_1, \ldots, a_k)$, where $I(a_1, \ldots, a_k)$ is a 1-simplex and (its centre)$\times P_k(a_1, \ldots, a_k) \subset G^4$ is nearly the natural inclusion $P_k(a_1, \ldots, a_k) \subset G^4$:

exactly the two embeddings are isotopic in G^4 by an isotopy which moves only a collar of the boundary of $P_k(a_1, \ldots, a_k)$. It is convenient to give a normal orientation to P^3 in N^4 (towards the exterior), to deduce from it the orientation of the 1-simplices $I(a_1, \ldots, a_k)$.

3.3. Construction of $g\colon G^4 \to B^2 \times D^2$

This g will be a smooth embedding which will reveal the structure of G^4. We choose, by recurrence, linear embeddings $I(a_1, \ldots, a_k) \subset (0, 1]$ conserving the orientation. To start, $I(\emptyset) \subset (0, 1]$ ends at 1. Suppose now these embeddings have been defined for all sequences of length $\leq k$. Then, we embed $I(a_1, \ldots, a_k, 0)$ and $I(a_1, \ldots, a_k, 1)$ respectively on the initial third and the final third of the interval $I(a_1, \ldots, a_k) \subset (0, 1]$.

The central third of $I(a_1, \ldots, a_k)$ is a closed interval that we may call $J(a_1, \ldots, a_k)$. The complement in $I(\emptyset)$ of all the open intervals $\mathring{J}(a_1, \ldots, a_k)$ is then a compact cantor set in $(0, 1]$.

On the other hand, we claim that the embeddings $i_k(a_1, \ldots, a_k)|\colon \partial_+ H_k(a_1, \ldots, a_k) \to B^2 \times \partial D^2$ define together a smooth map $i\colon P \to B^2 \times \partial D^2$. Let $\varphi\colon (0, 1] \times B^2 \times \partial D^2 \to B^2 \times D^2$ be the embedding $(t, x, y) \mapsto (x, ty)$. We will have the tendency to identify domain and codomain by φ.

We define $g\colon G^4 \to B^2 \times D^2$ on $I(a_1, \ldots, a_k) \times P_k(a_1, \ldots, a_k)$ by the rule that $(t, x) \mapsto \varphi(t, i(x))$. In order that definition makes sense, we have to first adjust, by isotopy, the trivialisation given by the I-fibres $I(a_1, \ldots, a_k) \times P_k(a_1, \ldots, a_k)$ in $(G^4, \mathcal{I})$, a routine task that is left to the reader.

3.4. Construction of $g_0\colon G_0^4 \to B^2 \times D^2$

Let G_0^4 be the union of G^4 and a small collar neighbourhood C^4 of $\partial_- N$ in N that respects δG^4 (see the figure for Section 3.2) Let us extend g to an embedding $g_0\colon G_0^4 \to B^2 \times D^2$. By uniqueness of collars, we can arrange g and g_0 so that g_0 sends $C^4 - \mathring{G}^4$ to $(B^2 - \lambda B^2) \times \mu D^2$, where $\lambda \in (0, 1]$ is near to 1 and μ to the initial point of $I(\emptyset)$. This completes the construction of $g_0\colon G_0^4 \to B^2 \times D^2$. Looking near g_0 and its image, we will claim that we have completely described the closure $\overline{G_0^4}$ of G_0^4 in $\widehat{N}^4$.

3.5. The image $g_0(G_0^4) \subset B^2 \times D^2$

Some notations again (see the two figures below). $T(a_1, \ldots, a_k) \equiv T_k(a_1, \ldots, a_k) = \partial_+ X(a_1, \ldots, a_k)$, a multi-solid torus $\subset B^2 \times \partial D^2$. $T_*(a_1, \ldots, a_k) = \varphi(J(a_1, \ldots, a_k) \times T(a_1, \ldots, a_k)) \subset B^2 \times \mathring{D}^2$, a radially thickened copy of $T(a_1, \ldots, a_k)$, called a *hole*. $B_* = \lambda B^2 \times \mu D^2$ (see definition of g_0), called the *central hole*. $F_k = \cup\{\varphi(I(a_1, \ldots, a_{k-1}) \times T(a_1, \ldots, a_k)) \mid k \text{ fixed}\}$. The frontier δF_k, $k \geq 2$, are indicated in dashed lines in the right hand figure below. $(B^2 \times D^2)_0 = B^2 \times D^2 - \mathring{B}_* - \cup\{\mathring{T}_*(a_1, \ldots, a_k)\}$, called the *holed standard handle*. $W_0 = \cap_k F_k$, a compactum in $(B^2 \times D^2)_0$. With these notations, we claim that the image $g_0(G_0^4)$ is $(B^2 \times D^2)_0 - W_0$.

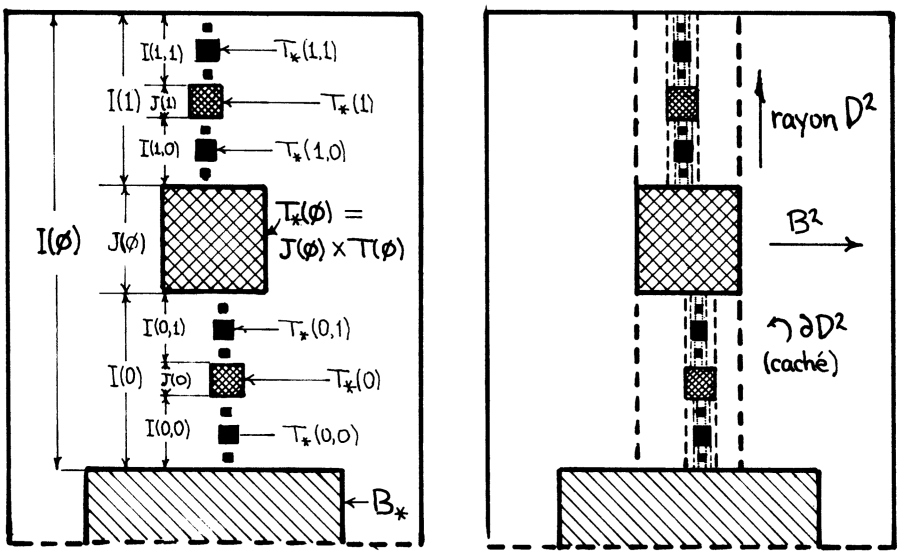


FIGURE 13.

3.6. The main diagram

The commutative diagram below gives an overview of the construction to come. The elements will be constructed in order $\mathcal{W}_0$, g_1, $\mathcal{D}'$, g_2, $\mathcal{D}$, g_3, $\mathcal{D}_+$, f. The proof that $(B^2 \times \mathring{D}^2)/\mathcal{D}_+$ is homeomorphic to $B^2 \times \mathring{D}^2$ (by the methods of Bing) is reserved to Section 4. The proof that f is approximable by homeomorphisms is postponed to Section 5.

$$\begin{tikzcd}
G_0^4 \arrow[r, "g_0"] \arrow[d, hook] & (B^2 \times D^2)_0 \arrow[d, "q"', "\mathrm{quot}"] \\
\bar{G}_0^4 \arrow[r, "g_1", "\approx"'] \arrow[d, hook] & (B^2 \times D^2)_0/\mathcal{W}_0 \arrow[d, "q_1"', "\mathrm{quot}"] \\
\hat{N}^4 \arrow[r, "g_2"] & (B^2 \times D^2)/\mathcal{D}' \\
M^4 \arrow[u, hook] \arrow[r, "g_3"] \arrow[dr, "f"'] & (B^2 \times \mathring{D}^2)/\mathcal{D} \arrow[u, hook] \arrow[d, "q_3"', "\mathrm{quot}"] \\
& (B^2 \times \mathring{D}^2)/\mathcal{D}_+
\end{tikzcd}$$

3.7. Construction of $\mathcal{W}_0$ and g_1

$\mathcal{W}_0$ is the decomposition of the compact set $(B^2 \times D^2)_0$, where non-degenerated elements are the connected components W of the compact set $W_0 \subset (B^2 \times D^2)_0$. Each $W \in \mathcal{W}_0$ is a Whitehead compactum in a single level $\varphi(t \times B^2 \times \partial D^2)$. We check naively that the inclusion $(B^2 \times D^2)_0 - W_0 \to (B^2 \times D^2)_0/\mathcal{W}_0$ induces a homeomorphism $\widehat{(B^2 \times D^2)_0 - W_0} \to (B^2 \times D^2)_0/\mathcal{W}_0$. We already know that $\widehat{G}_0$ is identified with $\overline{G}_0 \subset \widehat{N}$. We define the homeomorphism g_1 as a composition of homeomorphisms:

$$g_1 \colon \overline{G}_0 \longrightarrow \widehat{G}_0 \xrightarrow{\widehat{g}_0} \widehat{(B^2 \times D^2)_0 - W_0} \longrightarrow (B^2 \times D^2)_0/\mathcal{W}_0.$$

3.8. Construction of $\mathcal{D}'$ and g_2

Let $\mathcal{D}'$ be the decomposition of $B^2 \times D^2$ given by the B_*, $T_*(\alpha)$ (α can be any finite dyadic sequence), and the elements of $\mathcal{W}$ which are disjoint from them. To define $g_2\colon \widehat{N} \to (B^2 \times D^2)/\mathcal{D}'$, we must extend $q_1 g_1\colon \overline{G}_0 \to (B^2 \times D^2)/\mathcal{D}'$ to each connected component Y of $\widehat{N} - \overline{G}_0$. Its frontier δY is identified by g_1 to the quotient in $(B^2 \times D^2)_0/\mathcal{W}_0$, either of ∂B_*, or of a boundary of a connected component of a hole $T_*(a_1, \ldots, a_k)$. By definition, $g_2(Y)$ is the image in $(B^2 \times D^2)/\mathcal{D}$ of this boundary. It is easy to check the continuity of g_2.

Next, g_3 and $\mathcal{D}$ in the main diagram are defined by restriction. The design G^4 has led as inexorably to define $g_3\colon M^4 \to B^2 \times \mathring{D}^2/\mathcal{D}$, which compares the open Casson handle M^4 with a very explicit quotient of the open handle $B^2 \times \mathring{D}^2$.

The decomposition $\mathcal{D}$ which specifies this quotient has non-cellular elements, that is the holes $T_*(a_1, \ldots, a_k)$, each of which has the homotopy type of a circle. Therefore the quotient map $B^2 \times \mathring{D}^2 \to B^2 \times \mathring{D}^2/\mathcal{D}$ is certainly not approximable by homeomorphisms. One can also check that the Cech cohomology $\check{H}^2$ of the quotient is of infinite type.

The construction of $\mathcal{D}_+$ below repairs this terrible defect; it will be constructed by hand; $\mathcal{D}_+$ will be less fine than $\mathcal{D}$, which will enable us to define $f = q_3 \circ g_3$ without effort.

3.9. Construction of $\mathcal{D}_+$

We set $W = W_0 \cap (B^2 \times \mathring{D}^2) = W_0 - (B^2 \times \partial D^2)$. Its connected components define a decomposition $\mathcal{W}$ of $B^2 \times \mathring{D}^2$. We have known how to show for fifty years that $B^2 \times \mathring{D}^2/\mathcal{W}$ is homeomorphic to $B^2 \times \mathring{D}^2$, see Section 4.

For the need of the next paragraph, the quotient $(B^2 \times \mathring{D}^2)/\mathcal{D}_+$ must be a quotient of $B^2 \times \mathring{D}^2/\mathcal{W}$ by a decomposition whose elements are the connected components of

$$\cup\{q(T_*(\alpha)) \cup E(\alpha) \mid \alpha \text{ a finite dyadic sequence}\}.$$

Here $\{E(\alpha)\}$ is a collection of disjoint, topologically *flat* multi-2-discs such that for each finite dyadic sequence α, the intersection $E(\alpha) \cap (\cup_{\alpha'} q(T_*(\alpha')))$ is:

(1) the boundary $\partial E(\alpha)$; and
(2) a multi-longitude of $\partial T_*(\alpha)$ far from W (each connected component of $q(T_*(\alpha)) \cup E(\alpha)$ is then contractible).

Moreover, we want that the diameter of the connected components of $E(a_1, \ldots, a_k)$ tends towards 0 (on each compact set) as $k \to \infty$. Section 4 does not demand any more than this and visibly, $\{E(\alpha)\}$ specifies $\mathcal{D}_+$.

The specification of $\{E(\alpha)\}$ is unfortunately tedious. $E(\alpha)$ will be the faithful image $q(D(\alpha))$ of a multi-disc in $B^2 \times \mathring{D}^2$. For fundamental group reasons, the multi-disc $D(\alpha)$ is obliged to meet W, but, to assure flatness of $q(D(\alpha))$ (proved in Section 4), it must be a very gently meeting, permitted by (7) and (8) of Construction 3.1.

We have $T_k = \cup_\alpha T_k(\alpha)$; conditions (6) and (7) of Construction 3.1 assure that T_k is a multi-solid torus of which certain connected components constitute $T_k(\alpha)$. We have $\cap_k T_k = p(W)$, which is a ramified Whitehead compactum in $B^2 \times \partial D^2$.

To start, we specify (simultaneously and independently) in $B^2 \times \partial D^2$, (topologically) immersed, locally flat discs $D'(\alpha)$ which will be the projection $p(D(\alpha)) = D'(\alpha)$. We assume easily the two properties (a) and (b), where (b) uses (8) of Construction 3.1.

(a) $D'(a_1, \ldots, a_k)$ is a disjoint union of immersed discs in T_{k-1}, with as their only singularities, an arc of double points for each, above $T_k(a_1, \ldots, a_k)$. The boundary $\partial D'(a_1, \ldots, a_k)$ is formed from one longitude of each connected component of $\partial T_k(a_1, \ldots, a_k)$. The double points of $D'(a_1, \ldots, a_k)$ are outside $\mathring{T}_k(a_1, \ldots, a_k)$.
(b) For each $l \geq k$, the intersection $\mathring{D}'(a_1, \ldots, a_k) \cap T_l$ is a multi-disc (embedded in $T_k(a_1, \ldots, a_k)$) of which each connected component D_0 is a meridional disc of T_l that meets the solid tori of the next generator ($T_{l+1/6}$ with our revised indexing of Change of Notation 2.4) ideally (see the left hand figure of Figure 10).

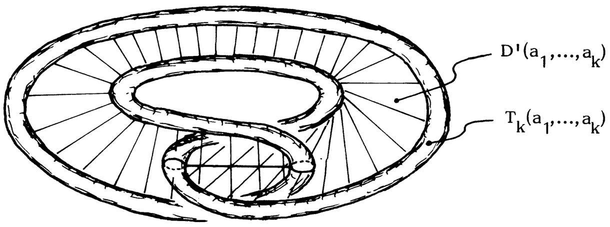
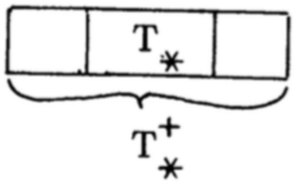


FIGURE 14.

By resolving the double points of $D'(\alpha)$, which we have to embed in $(0,1) \times B^2 \times \partial D^2 \subset B^2 \times D^2$, specifying the first coordinate by a convenient function $\rho(\alpha)\colon D(\alpha) \to (0,1)$.

We will embed a single $D(\alpha)$ at a time (following some chosen order). We embed first $D(a_1,\dots,a_k)$ closer and closer (by a secondary induction). Some notations:

$$T_*^+(a_1',\dots,a_l') = J(a_1',\dots,a_l') \times T(a_1',\dots,a_{l-1}'),$$
$$F_l^* = p^{-1}(p(F_l)) = (0,1) \times T_l,$$
$$W^+ = p^{-1}(p(W)) = (0,1) \times (\cap_k T_k).$$

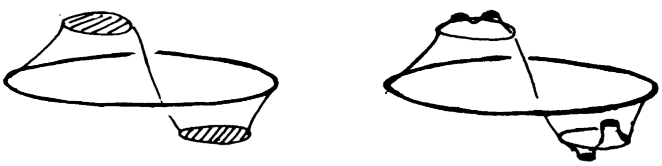


One can easily check that, for $D(a_1,\dots,a_k)$, the properties (c) and (d) for $l > k$, of which (d) for l is only provisional.

(c) $D(a_1,\dots,a_k)$ is embedded, is contained in $I(a_1,\dots,a_{k-1}) \times T(a_1,\dots,a_{k-1})$, and is disjoint from B_* and from $\cup\{T_*^+(\alpha') \mid \alpha' \neq (a_1,\dots,a_k)\}$. The boundary $\partial D(a_1,\dots,a_k)$ is in a signle level $t \times B^2 \times \partial D^2$, where $t \in \mathring{J}(a_1,\dots,a_k)$.

(d) Each connected component of the multi-disc $F_l^+ \cap D(a_1,\dots,a_k)$ is in a single level $t \times B^2 \times \partial D^2$; this level is disjoint from each box $T_*(\alpha')$, and does not contain any other connected component of $F_l^+ \cap D(a_1,\dots,a_k)$.

For $l = k$ and $k+1$, here are the illustrations of the graph of ρ in a simple case.



We observed that in pushing $D(a_1,\dots,a_k)$ vertically, as small as we want, and only on $\mathring{F}_l^+ \cap D(a_1,\dots,a_k)$, we can pass from (d) for l to (d) for $l+1$, without losing (c). Therefore, without losing (c), we can pass to the property:

(e) For each integer $l > k$, the connected component of the multi-disc $F_l^+ \cap D(a_1,\dots,a_k)$ project on as many disjoint intervals of radius in $(0,1)$.

This condition assures that, for all $W \in \mathcal{W}$, the intersection $W \cap D(a_1, \dots, a_k)$ is an intersection of discs (and so cellular). Therefore $q(D(a_1, \dots, a_k))$ is certainly a disc (compare Theorem 4.4) in Section 4, we will prove by hand that it is a *flat disc*. If, before $D(a_1, \dots, a_k)$, we have already defined (for the main induction) a finite collection of discs $D(\alpha_1), \dots, D(\alpha_n)$, we follow the same construction as above, always staying in a neighbourhood of $T_*^+(a_1, \dots, a_k)$ (guaranteed by (c)), disjoint from $D(\alpha_1) \cup \dots \cup D(\alpha_n)$ and for all elements of $\mathcal{W}$ that touch $D(\alpha_1) \cup \dots \cup D(\alpha_n)$.

Thus the family $\{D(\alpha)\}$ of *disjoint* 2-discs is defined by a double induction and verifies the properties (a), (b), (c) and (e) with $p(D(\alpha)) = D'(\alpha)$. Next $\{D(\alpha)\}$ defines $\mathcal{D}_+$ as already indicated. One easily checks all the properties wanted for $q(D(\alpha)) = E(\alpha)$ in $(B^2 \times \mathring{D}^2)/\mathcal{W}$, except local flatness of $E(\alpha)$ which is postponed to Section 4.

3.10. End of the proof that M is homeomorphic to $B^2 \times \mathring{D}^2$ (modulo Sections 4 and 5)

Accepting from Section 4 that $(B^2 \times \mathring{D}^2)/\mathcal{D}_+$ is homeomorphic to $B^2 \times \mathring{D}^2$, we show modulo Section 5 the approximability by homeomorphisms of $f\colon M^4 \to (B^2 \times \mathring{D}^2)/\mathcal{D}_+$ is the following fashion. We form the commutative diagram

$$\begin{tikzcd} \operatorname{int} M^4 \arrow[r, "f|"] \arrow[d, hook] & (\mathring{B}^2 \times \mathring{D}^2)/\mathcal{D}_+ \arrow[r, "\approx"] & \mathring{B}^2 \times \mathring{D}^2 \arrow[r, "\approx"] & S^4 - \infty \arrow[d, hook] \\ S^4 \arrow[rrr, dashed, "f_*"'] & & & S^4 \end{tikzcd}$$

where the inclusion $\operatorname{int} M \subset S^4$ exists since M embedds in $B^2 \times D^2$ (the experts also know that $\operatorname{int} M$ is diffeomorphic to $\mathbb{R}^4$ [Cas86]), and where $f_*(S^4 - \operatorname{int} M^4) = \infty$. Therefore,

$$S(f_*) = \{y \in S^4 \mid f_*^{-1}(y) \neq \text{ a point}\}$$

is visibly a contractible set.

Also $S(f_*)$ is nowhere dense. (Here is a proof. The restriction $f_*|$ is same as $q_3 \circ q_1 \circ g_1|\colon M \cap \overline{G}_0^4 \to (B^2 \times \mathring{D}^2)/\mathcal{D}_+$ is already surjective and $f_*^{-1}(S(f_*))$ is contained in the nowhere dense set of $M \cap \overline{G}_0^4$ given by $(\partial G_0) \cup (\text{ends of } G_0^4) \cup g_1^{-1}(\cup_\alpha E(\alpha))$.)

Therefore, according to Theorem 5.1, the map f_* is approxmiable by homeomorphisms. Next, by Proposition 4.2 (localisation principle), the restriction $\operatorname{int} M^4 \to (S^4 - \infty)$ is also approximable by homeomorphism. Finally, by Proposition 4.3 (globalisation principle), the map $f\colon M \to (B^2 \times \mathring{D}^2)/\mathcal{D}_+$ is approximable by homeomorphisms. Thus Theorem 2.2 is proved modulo Sections 4 and 5.

Remark. $\overline{S(f_*)} \subset S^4$ is in fact a compactum of dimension ≤ 1, because the union of a contractible set $S(f_*)$ with a set of dimension 0, that is the ends of G_0^4 which are not in the frontier of a connected component Y of $M^4 - G_0^4$. For reasons of cohomology, $\dim \overline{S(f_*)} \geq 1$; Therefore it is a compactum of dimension exactly 1.

4. Bing shrinking

We need to show that the space $B^2 \times \mathring{D}^2/\mathcal{D}_+$ defined in Section 3 is homeomorphic to $B^2 \times \mathring{D}^2$. The necessary techniques come from a series of articles of R. H. Bing from the 1950s (see especially [Bin52, Bin57, Bin59]), which made his reputation as a great virtuoso of geometric topology.

We consider proper surjective map $f\colon X \to Y$ between metrisable, locally compact spaces X, Y. Let $\mathcal{D} = \{f^{-1}(y) \mid y \in Y\}$ be the decomposition associted to f. When is f (strongly) *approximable by homeomorphisms*, in the sense that for all open covering $\mathcal{V}$ of Y, $\mathcal{V}$-neighbourhood

$$N(f, V) = \{g\colon X \longrightarrow Y \mid \text{for all } x \in X, \text{ there exists } V \in \mathcal{V} \text{ such that } f(x), g(x) \in V\}.$$

contains a homeomorphism?

Since f induces a homeomorphism $\varphi\colon X/\mathcal{D} \to Y$, we see easily that f is approximable by homeomorphisms if and only if one can find maps $g\colon X \to X$ such that $\mathcal{D} = \{g^{-1}(x) \mid x \in X\}$ and that

$f \circ g$ approxmates f (in effect, φ translates g into a homeomorphism $g' \colon Y \to X$). This observation makes the following theorem plausible.

Theorem 4.1 (Bing shrinking criterion)**.** *f is approximable by homeomorphism if and only if, for every coverings $\mathcal{U}$ of X and $\mathcal{V}$ of Y, there exists a homeomorphism $h \colon X \to X$ such that $h(\mathcal{D}) < \mathcal{U}$, and for all compact $D \in \mathcal{D}$, D and $h(D)$ are $f^{-1}(\mathcal{V})$-near in the sense that there exists an $f^{-1}(V) \in f^{-1}(\mathcal{V})$ that contains $D \cup h(D)$.*

We then say that $\mathcal{D}$ is *shrinkable*. We can show a proof by hand [Cha76], or by Baire category [Tor81], [Mor62] (the idea is to find a homeomorphism $h \colon X \to Y$ that converges towards g that determines $\mathcal{D}$). The proof also gives:

Remark. In Theorem 4.1, if h respect (or fix) a closed set $A \subset X$, then f is approximable by homeomorphisms that send A on $f(A)$ (or which coincide on A with f), and reciprocally.

Proposition 4.2 (Localisation principle)**.** *If $f \colon X \to Y$ is approximable by homeomorphisms and Y is a manifold (or Y satisfies the principle of deformability by homeomorphisms coming from* [EK71]*, $\mathcal{D}_1$ of* [Sie72b]*), then, for each open set V of Y, the restriction $f_V \colon f^{-1}(V) \to V$ of f is approximable by homeomorphisms.*

Proof (*indication*)*.* To approximate f_V, we combine (by the principal $\mathcal{D}_1$) a series of approximations of f; compare [Sie72b, Section 3.5]. I believe that this lemma is not in the literature because, for dimension $\neq 4$, we have stronger results [Sie72a, Edw77]. However, upon reflection, the complicated argument of [Sie72a] works. In each case that interests us, the reader will be able to find an *ad hoc* proof that is easier. □

Counterexample. This principle is false for X and Y are Cantor $\times [0,1] = 2^{\mathbb{N}} \times [0,1]$, and $f = g \times \mathrm{id}_{[0,1]}$ where $g(1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$, $g(0, a_2, a_3, \ldots) = (0, 0, 0, \ldots)$.

Proposition 4.3 (Globalisation principle)**.** *Let $f \colon X \to Y$ be a proper map such that, for an open set $V \subset Y$, the restriction $f_V \colon f^{-1}(V) \to V$ is approximable by homeomorphisms. Then, f is approximable by proper maps g such that*

(1) *$g^{-1}(V) = f^{-1}(V)$,*
(2) *$g_V \colon g^{-1}(V) \to V$ is homeomorphism, and*
(3) *$g = f$ on $X - f^{-1}(V)$.*

This principle is easy to establish, because if $\mathcal{V}$ is the covering of V by open balls centred on $y \in V$ and of radius $\inf\{d(y,z) \mid z \in Y - V\}$, then every map $\gamma \colon f^{-1}(V) \to V$ that is in $N(f_V, \mathcal{V})$, extends by f to a map $g \colon X \to Y$. In the very special case that $\mathcal{D}$ is $\pi_0(K)$ for a compact set $K \subset X$, the Bing shrinking criterion simplifies as follows: (Then, $\mathcal{D}$ consists of connected components of K and the image of K in $X/\mathcal{D}$ is 0-dimensional and is identified with $\pi_0(K)$.)

Theorem 4.4 (Criterion)**.** *Under these conditions, $\mathcal{D}$ is shrinkable if for all $\epsilon > 0$ and for all open $\mathcal{D}$-saturated U of X such that $U \cap K$ is compact, there is a homeomorphism $h \colon X \to X$ with support in U (respectively $A \subset X$) such that $h(K \cap U)$ is expressed in a finite disjoint union of compact sets, each of diameter $< \epsilon$.*

This condition, modulo localisation principle (Proposition 4.2), is clearly necessary.

For all $\epsilon > 0$, one can consider $\mathcal{D}_\epsilon = \{D \in \mathcal{D} \mid \operatorname{diam} D \geq \epsilon\}$. We say that $\cup_{D \in \mathcal{D}_\epsilon} D$ is a closed subset of X. Here is a remarkable but disturbing example where $\mathcal{D}$ is null, $\mathcal{D}_\epsilon$ is shrinkable for any $\epsilon > 0$, but $\mathcal{D}$ is not shrinkable. The elements of $\mathcal{D}$ are the connected components of a compact set $X = \cap_n F_n$ where F_0 and F_1 are as illustrated. This image is suitably replicated in each solid torus; F_n is then 2^n solid tori. Each $D \in \mathcal{D}$ is clearly cellular, hence $\mathcal{D}_\epsilon$ is shrinkable by Lemma 5.2. But, with the help of cyclic covers, one can check that $\mathcal{D}$ is not shrinkable (see [Bin62, AB67]).

There are thankfully properties of individual elements, a little stronger than cellularity, which discards this sort of example. For a compact $A \subset X$, we consider the property $\mathcal{R}(X, A)$: For each $\epsilon > 0$, for every null decomposition $\mathcal{D}$ of X containing A, and for all neighbourhoods U of A, there is a map $f \colon X \to X$ with support in U that shrinks at least A, (that is, $f(A)$ is a point and $f|_U \colon U \to U$

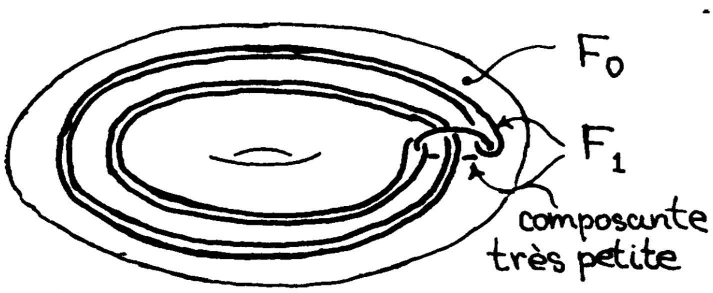


FIGURE 15.

is approximable by homeomorphisms), such that, for all $D \in \mathcal{D}$, $\operatorname{diam} f(D) \leq \max(\operatorname{diam} D, \epsilon)$. If $\mathcal{D}$ is fixed in advance, we call the (weaker) property $\mathcal{R}(X, A; \mathcal{D})$.

Observation. For every neighbourhood U of A, we have $\mathcal{R}(X, A)$ is equivalent to $\mathcal{R}(U, A)$. Moreover, $\mathcal{R}(X, A)$ is independent of the metric.

Proposition 4.5. *If $\mathcal{D}$ is null, and $\mathcal{R}(X, D; \mathcal{D})$ is satisfied for all $D \in \mathcal{D}$, then $\mathcal{D}$ is shrinkable.*

Proof. The proof is an edifying exercise. □

Proposition 4.6. *$\mathcal{R}(X, A)$ is satisfied if A is a topological flat disc of any codimension in the interior of the manifold.*

Proof of Proposition 4.6. This is $\mathcal{R}(\mathbb{R}^n, B^k)$ for $k \leq n$. The proof of $\mathcal{R}(\mathbb{R}^2, B^1)$ which is indicated by Figure 16.

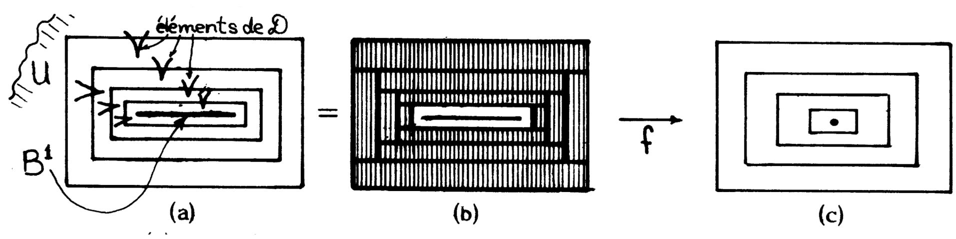


FIGURE 16.

In (a), every element of $\mathcal{D}$ that meets the big rectangle has already diameter $< \epsilon/4$; if $D \in \mathcal{D}$ meets a gap between successive rectangles, it is disjoint from the rectangle after. We set $f(B^1) = 0$, and $f = \mathrm{id}$ outside the biggest rectangle (which is in U); f is linear on each vertical interval in a rectangle of (b) and also linear on each 1-cell of the rectangular cellulation in (b) of (big rectangle$-B^1$); Moreover, $p \circ f = p$ where p is the projection to the y-axis (the $\mathbb{R}^{n-k}$ normal to B^k); Finally, the size of the image of each of the *vertical* rectangle is $< \epsilon/4$. □

We consider the Whitehead pair $(B^2 \times S^1, j(B^2 \times S^1)) = (T, T')$, and the *thickened* pair $(\mathbb{R} \times T, [0,1] \times T')$.

Lemma 4.7. *For $\epsilon > 0$, there exists an isotopy h_t $(t \in [0,1])$ of $\mathrm{id}|_{\mathbb{R}\times T}$ with compact support in $(-\epsilon, 1+\epsilon) \times \operatorname{int} T$ such that, we have $\operatorname{diam}(h_1(t \times T')) < \epsilon$ and $h_1(t \times T') \subset [t-\epsilon, t+\epsilon] \times T$ for all $t \in [0,1]$.*

Idea of a proof of Lemma 4.7. It is suggested by Figure 18. □

By this lemma, one can shrink many decompositions related to Whitehead compacta. For example, let $\mathcal{W} \subset \mathbb{R}^3$ be a Whitehead compactum and let $\mathcal{D} = \{t \in W \mid t \in [0,1], W \in \mathcal{W}\}$ be the decomposition $I \times \mathcal{W}$ of $\mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$. Then $\mathcal{D}$ is shrinkable by Lemma 4.7 applied to the solid tori

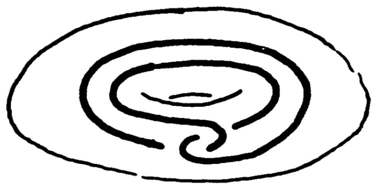


FIGURE 17.

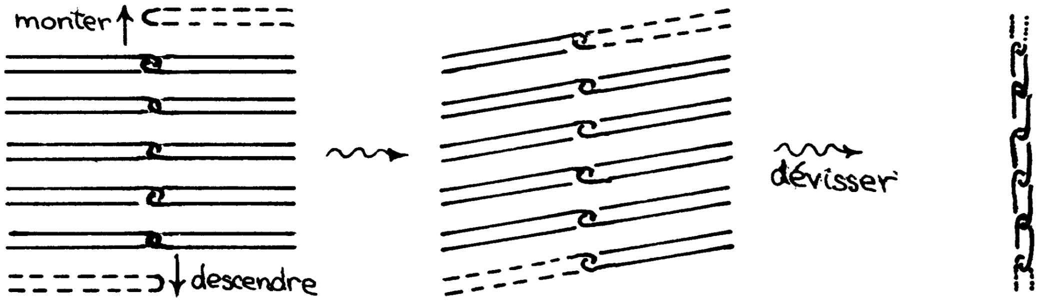


FIGURE 18.

$T, T', T'', \ldots$ of intersection $\mathcal{W}$. Therefore $\mathbb{R}^4/\mathcal{D}$ is homeomorphic to $\mathbb{R}^4$. Moreover, by Proposition 4.2 (localisation principle), we have $(0,1) \times \mathbb{R}^3/\mathcal{W}$ is homeomorphic to $(0,1) \times \mathbb{R}^3$. Hence we have the following celebrated fact.

Theorem 4.8 (Celebrated fact [AR65])**.** $\mathbb{R} \times (\mathbb{R}^3/\mathcal{W}) = \mathbb{R}^4$.

This is a result of Andrews and Rubin [AR65] in 1965, proved after analogous results, but more difficult, of Bing [Bin57] in 1959, which is a curious anachronism. There is a good explanation! A. Shapiro, at the time when he succeeded in turning S^2 inside out in S^3 by a regular homotopy, compare [FM80], had also established Theorem 4.8. In any cases, Bing tells me that D. Montgommery had communicated to him this claim without being able himself to apply it to more than an easier argument (see Lemma 4.9) showing $\mathbb{R} \times (S^3 - \mathcal{W})$ is homeomorphic to $\mathbb{R}^4$, compare [Bin59]. How the putative proof of Shapiro from the 50s seems to have disappeared without a trace.

To establish the flatness of discs $\{E(\alpha)\}$ constructed in Section 3.9, we will also need a lemma that is easier than Lemma 4.7, treating again the Whitehead pair (T, T'). Let D be a meridional disc of T that cuts T' transversally in 2 discs. Thus:

FIGURE 19.

Lemma 4.9. *With this data, we can find in* $\mathbb{R} \times T$ *a topological* 4*-ball* B*, such that* $\operatorname{int} B \supset [0,1] \times T'$ *and* $B \cap (\mathbb{R} \times D)$ *is an equatorial* 3*-ball of the form* (interval) $\times D_0 \subset \mathbb{R} \times D$*.*

Proof of Lemma 4.9. This has nothing to do with the proof of Lemma 4.7! We find B easily from a 2-disc immersed in T like in Figure 20 (compare Section 3.9). $\square$

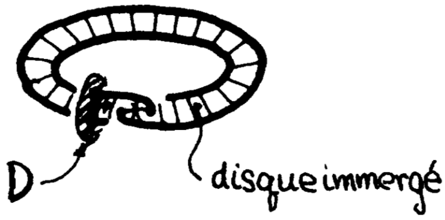


FIGURE 20.

To establish that $(B^2 \times \mathring{D}^2)/\mathcal{D}_+$ is homeomorphic to $B^2 \times \mathring{D}^2$, we will now use the construction of Section 3.

Proposition 4.10. *The decomposition $\mathcal{W}$ of $B^2 \times \mathring{D}^2$ is shrinkable.*

Proof of Proposition 4.10. We apply Theorem 4.4, Lemma 4.7 (or Lemma 4.9, without exploiting the last condition of Lemma 4.9). For this, it is convenient to remark first that for all open $\mathcal{W}$-saturated U in $B^2 \times D^2$, $W \cap U$ is contained in an open subset of U that is a disjoint union of open sets of the form $\mathring{I}' \times \mathring{T}(a_1, \dots, a_k)$, where I' is an interval. $\square$

Our next goal is the flatness of the discs $E(\alpha) = q(D(\alpha)) \subset (B^2 \times \mathring{D}^2)/\mathcal{W}$. Let $\mathcal{W}(\alpha) = \{w \in \mathcal{W} \mid w \cap D(\alpha) \neq \emptyset\}$, and let $W(\alpha) = \cup \mathcal{W}(\alpha)$.

Proposition 4.11. *$\mathcal{W}(\alpha)$ is shrinkable respecting $D(\alpha)$. Therefore, the quotient $q_\alpha(D(\alpha))$ of $D(\alpha)$ is flat in $(B^2 \times \mathring{D}^2)/\mathcal{W}(\alpha)$.*

Proof of Proposition 4.11. We apply Lemma 4.9 and the relative criteria (Theorem 4.4). For every open $\mathcal{W}_\alpha$-saturated U of $B^2 \times \mathring{D}^2$, the intersection $W_\alpha \cap U$ is trivially contained in an open set which, for some integer l, is a disjoint union of open sets of the form $\mathring{I}' \times \mathring{T}' \subset U$, where T' is a connected component of multiple solid tori $T_l(b_1, \dots, b_l)$ and i' is an interval.

The condition (d) of Section 3.9 allows us to choose these sets so that in addition, for each:

(∗) $D(\alpha) \cap (I' \times T')$ is a single 2-disc, which is projected onto a meridional disc D of T' which is also a connected component of $D'(\alpha) \cap T'$, see Section 3.9.

By the condition (b) of Section 3.9 the meridional disc D ideally chopped off $T_{l+1/6} \cap T'$, so Lemma 4.9 gives us disjoint 4-balls $B_1, \dots, B_s$ in $\mathring{I}' \times \mathring{T}'$, such that

(1) each intersection $B_i \cap D(\alpha)$ is a diametral 2-disc and not knotted in B_i, and
(2) $\mathring{B}_1 \cup \cdots \cup \mathring{B}_s$ contains the compact set $W^+ \cap (\mathring{I}' \times \mathring{T}') \supset W_\alpha \cap (\mathring{I}' \times \mathring{T}')$.

For all compact K in $\mathring{B}_i$ and all $\epsilon > 0$, we can easily find a homeomorphism $h \colon B_i \to B_i$ with compact support which respects $\mathring{B}_i \cap D(\alpha)$ and such that $\operatorname{diam} h(K) < \epsilon$. The criteria of Theorem 4.4 (respecting $D(\alpha)$) is therefore satisfied. $\square$

Proposition 4.12. *$q(D(\alpha)) = E(\alpha)$ is flat in $(B^2 \times \mathring{D}^2)/\mathcal{W}$.*

Proof of Proposition 4.12. The open set $U_\alpha = (B^2 \times \mathring{D}^2) - (W_\alpha \cup D(\alpha))$ is clearly homeomorphic to $(B^2 \times \mathring{D}^2)/\mathcal{W}_\alpha - q_\alpha(D(\alpha))$ by q_α. Therefore, by Propositions 4.2 and 4.3, the quotient map $q'_\alpha \colon (B^2 \times \mathring{D}^2)/\mathcal{W}_\alpha \to (B^2 \times \mathring{D}^2)/\mathcal{W}$ is approximable by homeomorphisms fixing q'_α on the flat disc $q_\alpha(D(\alpha))$. Therefore, $q(D(\alpha)) = q'(\alpha) \circ q(\alpha)(D(\alpha))$ is flat. $\square$

We now propose to finish by showing that the quotient maps:

$$B^2 \times \mathring{D}^2 \xrightarrow{\approx} (B^2 \times \mathring{D}^2)/\mathcal{W} \xrightarrow{p_1} ((B^2 \times \mathring{D}^2)/\mathcal{W})/\{E(\alpha)\} \xrightarrow{p_2} (B^2 \times \mathring{D}^2)/\mathcal{D}_+$$

are approximable by homeomorphisms.

Proposition 4.13. *p_1 is approximable by homeomorphisms.*

Proof of Proposition 4.13. This follows from Propositions 4.12, 4.6 and 4.5. □

To approximate p_2 by homeomorphisms, we need few preparations. According to Propositions 4.13 and 4.10, there is a shrinking map $r\colon B^2 \times \mathring{D}^2 \to B^2 \times \mathring{D}^2$ inducing the same decomposition as the quotient map $((B^2 \times \mathring{D}^2)/\mathcal{W})/\{E(\alpha)\}$; we can identify the domain of p_2 with $B^2 \times \mathring{D}^2$ by r.

The decomposition $\mathcal{P}$ constitutes of the pre-images $p_2^{-1}(y) = \{\text{a point}\}$ is the countable collection of natural collection of connected components of holes $T_*(\alpha)$ and B_*, which now identify $r(T_*(\alpha))$ and $r(B_*) \subset B^2 \times \mathring{D}^2$. We observe that $\mathcal{P}$ is null. The quotient map $\lambda B^2 \times \mu D^2 = B_* \to rB_*$ shrinks the Whitehead compactum $\mathcal{W}(\partial B_*) = \{w \in \mathcal{W} \mid w \subset B_*\}$, and these compact sets lie in $\lambda B^2 \times \mu \partial D^2 \subset \partial B_*$.

Proposition 4.14. *$r(\partial B_*)$ has a bicollar neighbourhood V in $B^2 \times \mathring{D}^2$, that is, $(V, r(\partial B_*)) \approx (\mathbb{R}, 0) \times r(\partial B_*)$.*

This will results the following proposition.

Proposition 4.15. *The quotient of ∂B_* in $(B^2 \times \mathring{D}^2)/\mathcal{W}(\partial B_*)$ admits a bicollar neighbourhood.*

Proof of Proposition 4.15. This is equivalent to the existence of a bicollar neighbourhood in $(\mathbb{R} \times \partial B_*)/(0 \times \mathcal{W}(\partial B_*))$. However, by (slightly generalised) Theorem 4.8 and Propositions 4.2 and 4.3, the quotient map of the latter space on $(\mathbb{R} \times \partial B_*)/(\mathbb{R} \times \mathcal{W}(\partial B_*))$ is approximable by homeomorphisms, fixing the quotient of $0 \times \partial B_*$. □

Proof of Proposition 4.14. The map r is factorised into $r'' \circ r'$ where r' factors through $\mathcal{W}(\partial B_*)$. However, Proposition 4.15 ensures a bicollar neighbourhood of $r'(\partial B_*)$ in $(B^2 \times \mathring{D}^2)/\mathcal{W}(\partial B_*)$. Propositions 4.2 and 4.3 ensure that r'' is approximable by homeomorphisms fixing $r'(\partial B_*)$. Therefore, the pair $((B^2 \times \mathring{D}^2)/\mathcal{W}(\partial B_*), r'(\partial B_*))$ (with the bicollar) is homeomorphic to $(B^2 \times \mathring{D}^2, r(\partial B_*))$. □

Proposition 4.16. *$\mathcal{R}(B^2 \times \mathring{D}^2, r(B_*); \mathcal{P})$ is satisfied.*

Proof of Proposition 4.16. Given an open neighbourhood U of $r(B_*)$, there exists, by Proposition 4.14, a homeomorphism $h\colon B^2 \times \mathring{D}^2 \to B^2 \times \mathring{D}^2$ with compact support in a bicollar V of $r(\partial B_*)$ in U, such that $h(r(B_*)) \subset r(\mathring{B}_*)$. Since $r(\mathring{B}_*)$ is homeomorphic to $\mathbb{R}^4$, there exists a map g with support in $r(\mathring{B}_*)$ and approximable by homeomorphisms such that $g \circ h \circ r(B_*)$ is a point in $r(\mathring{B}_*)$. Let $f = g \circ h\colon B^2 \times \mathring{D}^2 \to B^2 \times \mathring{D}^2$. By uniform continuity on the compact support $F \subset r(B_*) \cup V$ of f, we know that, for a given $\epsilon > 0$, there exists $\delta > 0$ such that for all set $E \subset B^2 \times \mathring{D}^2$ of diameter less than δ, the diameter of $f(E)$ is less than ϵ. By Lemma 4.17, there exists a stretch homeomorphism $\theta\colon B^2 \times \mathring{D}^2$ fixing $r(B_*)$ and support in V such that, for all $P \in \mathcal{P}$ distinct from B_* such that $\theta(P) \cap F \neq \emptyset$, we have $\operatorname{diam} \theta(P) < \delta$. Then $f = f_0 \circ \theta$ satisfies $\mathcal{R}(B^2 \times \mathring{D}^2, B_*; \mathcal{P})$. □

Lemma 4.17 (Stretch lemma)**.** *Let l be a null decomposition $X \times [0, \infty)$ where X is compact and all elements of l is disjoint from $X \times 0$. For all $\epsilon > 0$, there exists a homeomorphism with compact support $\varphi\colon [0, \infty) \to [0, \infty)$ such that $\Phi = \varphi \times \mathrm{id}_X$ satisfies that for all $E \in l$, such that $\Phi(E) \cap (X \times [0, 1]) \neq \emptyset$, we have* $\operatorname{diam}(\Phi(E)) < \epsilon$.

Proof of Lemma 4.17 (indication). Figure 21 completes the proof. □

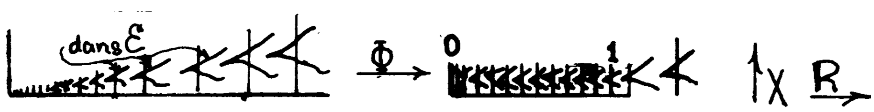


Figure 21.

All elements of $\mathcal{P}$ distinct from $r(B_*)$ is of the form $r(T'_*(\alpha))$ where $T'_*(\alpha)$ is a connected component of a torus $T_*(\alpha)$. Following the method of the proof of Proposition 4.16, we establish similarly the following proposition.

Proposition 4.18. *$\mathcal{R}(B^2 \times \mathring{D}^2, r(T'_*(\alpha)); \mathcal{P})$ is satisfied.*

Proof of Proposition 4.18 (indications). The quotient of $T'_*(\alpha) = J(\alpha) \times T'(\alpha)$, by the longitude $l(\alpha)$ that is in $D(\alpha)$, is a cone whose centre is the quotient of $l(\alpha)$, and the base is a solid torus. $\delta r(T'_*(\alpha)) - r(l(\alpha))$ has a bicollar neighbourhood in $B^2 \times \mathring{D}^2$, compare Proposition 4.13. The accumulation points of elements $P \neq$ a point of $\mathcal{P}$ are the *centre* $r(l(\alpha))$ and a compact set $r(W \cap \partial T'_*(\alpha))$ far from $r(l(\alpha))$. □

Proposition 4.19. *p_2 is approximable by homeomorphisms and hence $B^2 \times \mathring{D}^2/\mathcal{D}_+ \approx B^2 \times \mathring{D}^2$.*

Proof. Apply Propositions 4.18, 4.16 and 4.5. □

5. Freedman's approximation theorem

Theorem 5.1 (Freedman's approximation theorem)**.** *Suppose that X and Y are homeomorphic to the n-sphere. Let $f\colon X \to Y$ be a surjective, continuous map such that the singular set $S(f) = \{y \in Y \mid f^{-1}(y) \neq \text{a point}\}$ is nowhere dense and at most countable. Then, f can be approximated by homeomorphisms.*

Remark. For all dimensions $\neq 4$, there exist much stronger approximation theorems [Arm71, Sie70, Edw77]. Therefore, in dimension 4, the problem of generalising Theorem 5.1 clearly remains open.

In the case of $S(f)$ is finite, this theorem is well-known since it constitutes the essential part of the celebrated Schönflies theorem which was established around 1960 by B. Mazur, M. Brown and M. Morse.

Recall that a compact set A in a topological n-manifold M (without boundary) is *cellular* if each neighbourhood of A contains a neighbourhood which is homeomorphic to B^n.

Lemma 5.2. *Let A be a compact, cellular set in the interior* $\operatorname{int} M$ *of a manifold M. Then, the quotient map $q\colon M \to M/A$ can be approximated by homeomorphisms which are supported in an arbitrarily given neighbourhood of A.*

Compare the Bing shrinking criterion, Theorem 4.1 [Bro60]. A direct proof shrinks A gradually to a point.

Proof of Theorem 5.1 if $S(f)$ is a point. Let $y_0 = S(f)$ and $A = f^{-1}(y_0)$, we have that $X - A$ is homeomorphic to $\mathbb{R}^n$. Since X is homeomorphic to S^n, it follows that A is cellular in X (exercise). Then, we obtain approximations by applying Lemma 5.2. □

In the setting of Freedman's ideas, the case where $S(f)$ is n points, $n \geq 2$, is already as difficult as Theorem 5.1. However one can consult [Bro60, Dou61] for an easy proof. We recall the Schönflies theorem.

Theorem 5.3 (Schönflies theorem)**.** *Let Σ^{n-1} be a topologically embedded $(n-1)$-sphere in S^n such that there is a bicollar neighbourhood N of Σ in S^n, that is, (N, Σ) is homeomorphic to $(\Sigma \times [-1, 1], \Sigma \times 0)$. Then the closure of each of the two components of $S^n - \Sigma$ is homeomorphic to the n-ball B^n.*

Proof of Theorem 5.3 (starting from Theorem 5.1 for $S(f)$ consists of 2-points). Let X_1 and X_2 be two connected components of $S^n - \mathring{N}$ and W_1 and W_2 be the closures of connected components of $S^n - \Sigma^{n-1}$ containing X_1 and X_2, respectively. It is necessary to show that W_1 and W_2 are homeomorphic to B^n.

Shrinking X_1 and X_2, we obtain a quotient map

$$f\colon S^n \longrightarrow S^n/\{X_1, X_2\} \approx (\Sigma \times [-1, 1]/\{\Sigma \times 0, \Sigma \times 1\} \approx S^n$$

that is approximable by homeomorphisms from Theorem 5.1 (the case of $S(f)$ is two points). So X_1 and X_2 are cellular in S^n. Apply Lemma 5.2 to $X_i \subset \mathring{W}_i$, we deduce that

$$W_i \longrightarrow W_i/X_i \approx \Sigma \times [0, 1]/\{\Sigma \times 1\} \approx B^n$$

is approximable by homeomorphisms. □

Observation. The case of Theorem 5.3, where we know in advance that Σ bounds an n-ball in S^n, already arises from the case of Theorem 5.1 where $S(f) = \{1 \text{ point}\}$ proved above. Freedman uses this case.

To prove Theorem 5.1, Freedman introduced a nice trick of iterated replication of the approximation map, which vaguely reminds me of the arguments of Mazur [Maz59]. This trick leads us to leave the category of continuous maps and to instead work in the less familiar realm of closed relations. It was during the seventies that closed relations imposed themselves for the first time on geometric topology; they surfaced implicitly in a very original article by M. A. Stanko [Sta73] and have become essential since: I believe that it would be a herculean task to prove, without closed relations, the subsequent theorem of Ancel and Cannon [AC79] that any topological embedding $S^{n-1} \to S^n$, $n \geq 5$, can be approximated by locally flat embeddings.

Definition. A *closed relation* $R\colon X \to Y$ between metrisable spaces X and Y is a closed subset R of $X \times Y$. If $S\colon Y \to Z$ is a closed relation, the composition $S \circ R\colon X \to Z$ is

$$S \circ R = \{(x, z) \in X \times Z \mid \text{there is } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\},$$

which is also closed if Y is compact. Therefore the collection of closed relations between compact spaces is a category.

A continuous map $f\colon X \to Y$ gives a closed relation $\{(x, f(x)) \mid x \in X\}$ (the graph of f) which we still call f. Reciprocally, provided that Y is compact, a closed relation $R\colon X \to Y$ is the graph of a continuous function (which is uniquely determined) if $R \cap x \times Y$ is a point for all $x \in X$.

Remark. The natural function $[0, 1) \to \mathbb{R}/\mathbb{Z}$ is continuous and bijective; the inverse is discontinuous, but the graphs of two are closed.

By extending usual notions for continuous functions, for $A \subset X$ and $B \subset Y$, we have

(1) the *image* $R(A) = \{y \in Y \mid \text{there exists } x \in A \text{ such that } (x, y) \in R\}$,
(2) the *restriction* $R|_A\colon A \to Y$ is the closed subset $R \cap A \times Y$ in $A \times Y$,
(3) the *inverse* $R^{-1}\colon Y \to X$ such that $\{(y, x) \in Y \times X \mid (x, y) \in R\}$.

Remark. R^{-1} is the inverse of R in the categorical sense if and only if R is the graph of a bijection function (if and only if the categorical inverse exists).

To exploit an analogy between a function $X \to Y$ and a relation $R\colon X \to Y$, we will at any time assimilate R to the function that associates for each point $x \in X$ to a subset $R(x) \subset Y$.

Proof of Theorem 5.1. Any submanifold of codimension 0 that is introduced will be assumed to be topological and locally flat. Let N be a neighbourhood of f in $X \times Y$. The theorem asserts that there exists a homeomorphism $H\colon X \to Y$ such that $H \subset N$.

By removing a small n-ball $D \subset Y - \overline{S(f)}$ from Y and removing its pre-image $f^{-1}(D)$ from X, we see that it is permissible to adopt the following.

Theorem 5.4 (Change of data)**.** *Suppose X and Y are homeomorphic to B^n rather than S^n. Let $f\colon X \to Y$ be a surjective, continuous map such that the singular set $S(f) = \{y \in Y \mid f^{-1}(y) \neq \text{a point}\}$ is nowhere dense and at most countable and $S(f) \subset \operatorname{int} Y$. Then f can be approximated by homeomorphisms.*

It is easy to see that Theorem 5.4 implies Theorem 5.1 using the special case of Theorem 5.3 (Schönflies theorem) where Σ^{n-1} bounds a ball (see observation after Theorem 5.3).

The first step of an inductive construction of H is to apply the following proposition to the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f"'] & & Y \arrow[dl, "\mathrm{id}"] \\ & Y & \end{tikzcd}$$

Moreover, the neighbourhood N of Proposition 5.5 becomes N the above; and L becomes Y.

Suppose that X and Y are homeomorphic to B^n. A relation $R\colon X \to Y$ is called *good* if it is closed, and satisfying the following conditions.

(1) $R \subset X \times Y$ is projected onto X and onto Y.

(2) $R(x)$ is not a singleton set for at most countably many points in X and these exceptional points constitute a nowhere dense set contained in $\operatorname{int} X$. The same holds for R^{-1}.

It is said that a good relation $R'\colon X \to Y$ is *finer than* R if $R' \subset R \subset X \times Y$.

Proposition 5.5. *Given a triangle of good relations* (*which is eventually commutative*)

$$\begin{tikzcd} X \arrow[rr, "R"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

where X, Y and Z are homeomorphic to B^n, and f, g are in addition continuous functions; *a neighbourhood N of R in $X \times Y$*; *and $L \subset Z$ an open subset* (*called the gap*). *We impose the following conditions.*

(*a*) $R \subset (f^{-1}(\overline{L}) \times g^{-1}(\overline{L})) \cup (f^{-1}(Z-L) \times g^{-1}(Z-L))$; *it is inevitable if the triangle switches.*

(*b*) $R = g^{-1} \circ f$ *on* $f^{-1}(\overline{L})$.

(*c*) *R is given by the intersection graph of a homeomorphism* $f^{-1}(Z-L) \to g^{-1}(Z-L)$.

(*d*) *The singular sets $S(f)$ and $S(g)$ are separated on L, that is, there are two open disjoint sets U and V which contain $S(f) \cap L$ and $S(g) \cap L$, respectively.*

Then, for all $\epsilon > 0$, we can modify the three data g, R, L in g_*, R_*, L_* so that in addition to the same conditions above (*with g_*, R_*, L_* instead of g, R, L*), *we have $R_* = R$ on $f^{-1}(Z-L)$, $L_* \subset L$, and for all $y \in Y$,* $\operatorname{diam} R_*^{-1}(y) < \epsilon$.

Complement. *There exists a neighbourhood $N_* \subset N$ of R_* in $X \times Y$ such that we have* $\operatorname{diam}(N_*^{-1}(y)) < \epsilon$ *for all $y \in Y$.*

Proof of Complement. If the conclusion is false, then there are two sequences of points of $X \times Y$, say (x_k, y_k), (x_k, y'_k), $k = 1, 2, 3, \ldots$, which converge in compact R_* and such that $d(y_k, y'_k) \geq \epsilon$. By compactness of $X \times Y$, we can arrange that the sequences x_k, y_k and y'_k converge to x, y and y', respectively. Then, (x, y) and (x, y') belong to compact R_*, but $d(y, y') \geq \epsilon$, which is a contradiction. □

Proposition 5.5 (with Complement) will be used as a machine that swallows the data f, g, R, L, N, ϵ and manufactures f, g_*, R_*, L_*, N_*.

Continuity of the homeomorphism H is proved by assuming Proposition 5.5. For $k \geq 1$, the k-th step constructs a triangle (where Z is a copy of Y):

$$\begin{tikzcd} X \arrow[rr, "R_k"] \arrow[dr, "f_k"'] & & Y \arrow[dl, "g_k"] \\ & Z & \end{tikzcd}$$

a submanifold $L_k \subset Z$, and a neighbourhood N_k of R_k in $X \times Y$ such that f_k, g_k, R_k, L_k, N_k satisfy the conditions imposed on f, g, R, L, N in Proposition 5.5. The first step is already specified. Proposition 5.5 gives f_1, g_1, R_1, L_1, N_1 from f, id, f, Y, N, 1.

Suppose that the k-th triangle is constructed and we construct the $k+1$-th triangle.

(a) If k is odd, then Proposition 5.5 gives g_{k+1}, f_{k+1}, R_{k+1}^{-1}, L_{k+1}, N_{k+1}^{-1} from g_k, f_k, R_k^{-1}, L_k, N_k^{-1}, $1/k$. In brief, we apply Proposition 5.5 to the reverse triangle

$$\begin{tikzcd} Y \arrow[rr, "R_k^{-1}"] \arrow[dr, "g_k"'] & & X \arrow[dl, "f_k"] \\ & Z & \end{tikzcd}$$

(b) If k is even, then it is same as the first step: Proposition 5.5 gives f_{k+1}, g_{k+1}, R_{k+1}, L_{k+1}, N_{k+1} from f_k, g_k, R_k, L_k, N_k, $1/k$.

By induction, we have $N \supset N_1 \supset N_2 \supset \cdots$. We define $H = \cap_k N_k$. Then, H is a homeomorphism since, for all x, we have $\operatorname{diam} H(x) \leq \operatorname{diam} N_k(x) \leq 1/k$, for all even k, and $\operatorname{diam} H^{-1}(x) \leq \operatorname{diam} N_k^{-1}(x) \leq 1/k$, for all odd k. This homeomorphism H in the neighbourhood N of f completes the proof of Theorem 5.1 assuming Proposition 5.5. □

Proof of Proposition 5.5. To explain the essential idea of Freedman, the reader should read the proof to (re)prove that a surjection $f\colon B^n \to B^n$ such that $S(f) = \{\text{a point}\} \subset \operatorname{int} B^n$ is approximable by homeomorphisms (for this, we set $f = R$ and $g = \mathrm{id}$). Then, it should be noted that as soon as $S(f) = \{k \text{ points}\} \subset \operatorname{int} B^n$, the same argument leads us to approximate f by relations which crush nothing, but which blow up $k(k-1)$ points.

Consider the pre-images $R^{-1}(y)$, $y \in Y$, of diameter $\geq \epsilon$, that we want to eliminate. According to (a), (b) and (c), these sets constitute the pre-image by f of the set $(S_\epsilon(f) \cap L) \subset Z$, where $S_\epsilon(f) = \{z \in Z \mid \operatorname{diam} f^{-1}(z) \geq \epsilon\}$, which will allow us to follow the case in Z. Note that $S_\epsilon(f)$ is compact although, typically, $S(f)$ is not. For example, $S_\epsilon(f)$ is finite in the case of interest Freedman (see Section 4).

Lemma 5.6 (General position)**.** *In the interior of a compact topological manifold M, let A and B be two countable sets and nowhere dense. Then, there exists a small automorphism θ of M fixing all points of ∂M such that $\theta(A)$ and B are separated, that is, contained in disjoint open sets.*

Proof of Lemma 5.6. Consider the space $\operatorname{Aut}(M, \partial M)$ of automorphisms of M fixing ∂M, provided with the complete metric $\sup(d(f,g), d(f^{-1}, g^{-1}))$ where d is the uniform convergence metric. In $\operatorname{Aut}(M, \partial M)$, the set of automorphisms θ, such that the first k points A_k of A and B_k of B satisfying $\theta(A_k) \cap \overline{B} = \emptyset = \theta(\overline{A}) \cap B_k$, constitute an open subset $U_k \subset \operatorname{Aut}(M, \partial M)$ everywhere dense in $\operatorname{Aut}(M, \partial M)$, because $\overline{A}$ and $\overline{B}$ are closed, nowhere dense in M.

Then, famous Baire category theorem asserts that the countable intersection $\cap_k U_k$ is everywhere dense in $\operatorname{Aut}(M, \partial M)$. Note that $\cap_k U_k$ is the set of θ in $\operatorname{Aut}(M, \partial M)$ such that $\theta(A) \cap \overline{B} = \emptyset = \theta(\overline{A}) \cap B$. But, for X_1, X_2 in a metrisable M, the condition $X_1 \cap \overline{X}_2 = \emptyset = \overline{X}_1 \cap X_2$ leads the separation of X_1 and X_2 in M. In effect, seen in the open subset $M - (\overline{X}_1 \cap \overline{X}_2)$ of M, the set $\overline{X}_1 - (\overline{X}_1 \cap \overline{X}_2)$ and $(\overline{X}_1 \cap \overline{X}_2)$ are always disjoint, closed and hence separated. The mentioned condition ensures that they contain respectively X_1 and X_2. □

Claim 5.7 (Trivial if $S_\epsilon(f)$ is finite)**.** *There exists a finite union B_+ of disjoint n-balls in L satisfies the following conditions*:

1. $S_\epsilon(f) \cap L \subset \mathring{B}_+$.
2. $S(g) \cap B_+ = \emptyset$.
3. *Each connected component B'_+ of B_+ is small in the sense that $(f^{-1}(B'_+)) \times (g^{-1}(B'_+)) \subset N$, and standard in the sense that $Z - \operatorname{int} B'_+$ is homeomorphic to $S^{n-1} \times [0,1]$.*

Proof of Claim 5.7. Identify Z with $B^n \subset \mathbb{R}^n$ to give L an affine linear structure. Let K be a compact neighbourhood of compact $S_\epsilon(f) \cap L$ which is a subpolyhedra of L and disjoint from $S(g)$, see Proposition 5.5(c). We subdivide K into a simplicial complex of which each simplex $\mathcal{L}$ is linear in L and so small such that $f^{-1}(\Delta) \cap g^{-1}(\Delta) \subset N$. Then (compare, the proof of Lemma 5.6), by a small perturbation (a translation if we want) of K in L, we disengage the $(n-1)$-skeleton $K^{(n-1)}$ from the compact countable $S_\epsilon(f)$, without harming the properties of K already established. Finally, B_+ is defined as K minus a small δ open neighbourhood of $K^{(n-1)}$ in $\mathbb{R}^n$. Each component B'_+ of B_+ is convex and in $\operatorname{int} Z = \mathring{B}^n$; therefore $Z - \mathring{B}'_+$ is homeomorphic to $S^{n-1} \times [0,1]$, by an elementary argument. □

In $\operatorname{int} B_+$, we choose now a union B of balls (one in each connected component of B_+), which still satisfies (1), (2), (3) and also

4. $S(f) \cap \partial B = \emptyset$.

We set $L_* = L - B$. For each connected component B'_+ of B_+, we are now modifying g and R above B'_+ to define g_* and R_*. These changes for the various connected components B'_+ are disjoint and independent. Therefore, it is enough to specify one. Moreover, in order to simplify the notations, we allow ourselves to specify this change only in the case B_+ is connected.

Let $c\colon Z \to B_+$ be a homeomorphism, called the compression, which fixes all points of B. (We remember that $Z - \mathring{B}$ is homeomorphic to $S^{n-1} \times [0,1]$ and $B_+ - \mathring{B}$.) We should modify c by composing with a homeomorphism θ of $B_+ - \mathring{B}$ fixing $\partial B_+ \cup \partial B$ given by Lemma 5.6, to assure that $S(f)$ and $c(S(f))$ are separated on the open $\mathring{B}_+ - B$.

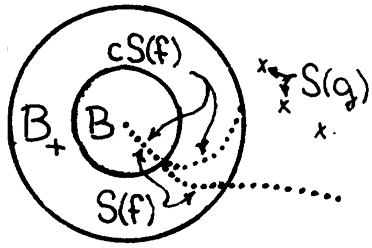


FIGURE 22.

Since $g^{-1}(B_+)$ is a ball in Y (in fact, g is a homeomorphism over B_+), we can also choose i so that $i|_{\partial X}$ is $(g^{-1} \circ c \circ f)|_{\partial X}$. We set

$$g_* = \begin{cases} g & \text{on } g^{-1}(Z - \mathring{B}_+), \\ c \circ f \circ i^{-1} & \text{on } g^{-1}(B_+). \end{cases}$$

On $g^{-1}(\partial B_+)$, g_* is well-defined since $g = c \circ f \circ (g^{-1} \circ c \circ f)^{-1}$ on $g^{-1}(\partial B_+)$. We set

$$R_* = \begin{cases} R & \text{on } f^{-1}(Z - \mathring{B}_+), \\ g^{-1} \circ f = (i \circ f^{-1} \circ c^{-1}) \circ f & \text{on } f^{-1}(B_+). \end{cases}$$

More precisely, on $f^{-1}(B_+)$, we specify

$$R_* = \begin{cases} (i \circ f^{-1} \circ c^{-1}) \circ f & \text{on } f^{-1}(B_+ - \mathring{B}), \\ i & \text{on } f^{-1}(B). \end{cases}$$

On $f^{-1}(\partial B)$, R_* is well-defined since c fixes all points of ∂B, and $S(f) \cap \partial B = \emptyset$.

We now specified the modification L_*, g_*, R_* of L, g, R claimed by Proposition 5.5. (We remark that if B_+ is a union of k balls rather than one, the modification is done in k disjoint and independent steps, each similar to the one just specified for connected B_+.)

Verifying the claimed properties for L_*, g_*, R_* is direct. (There are already manuscripts [Fre79, Anc81] which offer more details.) □

Remark 1. The system of the above formula, specifying g_* and R_*, hides geometry. We now try to reveal it by looking f and g respectively as fibrations φ and γ, of base Z, and variable fibre, which allows us to use the notion of fibre restriction. Let $\gamma_0 = \gamma - (\gamma|_{\mathring{B}_+})$. We form γ_* of $\varphi \sqcup \gamma_0$ by identifying $c|_{\partial Z}$ the sub-fibres (whose fibres are points) $\varphi|_{\partial Z}$ and $\gamma_0|_{\partial B_+}$. Then, we can identify the total space and the base of $\gamma_* = \varphi \cup \gamma_0$ to those of γ by an extension of $\gamma_0 \to \gamma$. More precisely, we use $(\mathrm{id}\,|_{Z - \mathring{B}_+}) \cup c$ between bases, which is the identity on $B \subset B_+$. Then, φ and $\gamma_*\colon Y \xrightarrow{g_*} Z$ are fibrations over Z naturally isomorphic on $B \cup L$, which defines a relation R_* and finer than the simple correspondence of fibres $g_*^{-1} \circ f$.

Remark 2. In the proof of Theorem 5.1, we can easily ensure that f_n and g_n converge towards f_∞ and g_∞, and such that $g_\infty \circ H = f_\infty$. Thus, as fibres, f_∞ and g_∞ are isomorphic. Moreover, each fibre of f_∞ or g_∞ is homeomorphic to a fibre of f. I point out that f_∞ and g_∞ remind me of the two infinite products of Mazur [Maz59] and that H reminds me of the famous *Eilenberg-Mazur swindle*, which completes the proof of Theorem 5.3 (weakened version) given in [Maz59].

Remark 3 (Following Remark 2)**.** If we want to avoid unnecessary complications in the structure of f_∞ and g_∞, it should be noted that in the definition of g_* above, we have the right to replace the map f which occurs by any good map $f'\colon X \to Y$ such that $f = f'$ on $f^{-1}(B)$. Then, for any use of Proposition 5.5 in the proof of Theorem 5.1, we find that one has the possibility of choosing for an alternative f' always a map isomorphic to to a map f given in Theorem 5.1. With this little refinement, the proof of Theorem 5.1 in the case $S(f) = \{2 \text{ points}\}$ is close to the argument of [Maz59]. In particular, $\overline{S(f_\infty)}$ and $\overline{S(g_\infty)}$ can be homeomorphic to $\mathbb{Z} \cup \{\infty, -\infty\}$.

References

[AB67] S. Armentrout and R. H. Bing, *A toroidal decomposition of E^3*, Fund. Math. **60** (1967), 81–87.

[AC79] F. D. Ancel and J. W. Cannon, *The locally flat approximation of cell-like embedding relations*, Ann. of Math. (2) **109** (1979), no. 1, 61–86.

[Anc81] F. Ancel, *Nowehere dense tame 0-dimensional decompositions of S^4 (an exposition of a theorem of mike freedman)*, manuscript, 1981.

[AR65] J. J. Andrews and L. Rubin, *Some spaces whose product with E^1 is E^4*, Bull. Amer. Math. Soc. **71** (1965), 675–677.

[Arm71] S. Armentrout, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, American Mathematical Society, Providence, R. I., 1971, Memoirs of the American Mathematical Society, No. 107.

[Bin52] R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) **56** (1952), 354–362.

[Bin57] R. H. Bing, *Upper semicontinuous decompositions of E^3*, Ann. of Math. (2) **65** (1957), 363–374.

[Bin59] R. H. Bing, *The cartesian product of a certain nonmanifold and a line is E^4*, Ann. of Math. (2) **70** (1959), 399–412.

[Bin62] ———, *Decompositions of E^3*, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall. Englewood Cliffs, N.J., 1962, pp. 5–21.

[Bro60] M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

[Bro72] W. Browder, *Surgery on simply-connected manifolds*, Springer-Verlag, New York-Heidelberg, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65.

[Cas86] A. Casson, *Three lectures on new-infinite constructions in 4-dimensional manifolds*, À la recherche de la topologie perdue, Progr. Math., vol. 62, Birkhäuser Boston, Boston, MA, 1986, With an appendix by L. Siebenmann, pp. 201–244.

[Cha76] T. Chapman, *Lectures on Hilbert cube manifolds*, American Mathematical Society, Providence, R. I., 1976, Expository lectures from the CBMS Regional Conference held at Guilford College, October 11-15, 1975, Regional Conference Series in Mathematics, No. 28.

[Dou61] A. Douady, *Plongements de sphères (the work of Mazur and Brown)*, Séminaire Bourbaki, Soc. Math. France, Paris, 1960–1961, pp. Exp. No. 205, 5–10.

[Edw75] R. D. Edwards, *Demension theory. I*, Geometric topology (Proc. Conf., Park City, Utah, 1974), Springer, Berlin, 1975, pp. 195–211. Lecture Notes in Math., Vol. 438.

[Edw77] ———, *Approximating certain cell-like maps by homeomorphisms*, manuscript, 1977.

[EK71] R. D. Edwards and R. Kirby, *Deformations of spaces of imbeddings*, Ann. Math. (2) **93** (1971), 63–88.

[FM80] G. Francis and B. Morin, *Arnold Shapiro's eversion of the sphere*, Math. Intelligencer **2** (1979/80), no. 4, 200–203.

[Fre79] M. H. Freedman, *A fake $S^3 \times \mathbf{R}$*, Ann. of Math. (2) **110** (1979), no. 1, 177–201.

[GS84] Robert E. Gompf and Sukhjit Singh, *On Freedman's reimbedding theorems*, Four-manifold theory (Durham, N.H., 1982), Contemp. Math., vol. 35, Amer. Math. Soc., Providence, RI, 1984, pp. 277–309.

[KM63] M. Kervaire and J. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537.

[KS77] R. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977, With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[Man80] R. Mandelbaum, *Four-dimensional topology: an introduction*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 1, 1–159.

[Maz59] B. Mazur, *On embeddings of spheres*, Bull. Amer. Math. Soc. **65** (1959), 59–65.

[MH73] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York-Heidelberg, 1973, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[Mil56] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. (2) **64** (1956), 399–405.

[Mil65] ———, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press, Princeton, N.J., 1965.

[Mor62] Claude Morlet, *Le lemme de Thom et les théorèmes de plongement de Whitney*, Séminaire Henri Cartan, 1961/62, Exp. 4–7, Secrétariat mathématique, Paris, 1961/1962, p. 5.

[MV75] A. Marin and Y. M. Visetti, *A general proof of Bing's shrinkability criterion*, Proc. Amer. Math. Soc. **53** (1975), no. 2, 501–507.

[Sie70] L. C. Siebenmann, *Infinite simple homotopy types*, Nederl. Akad. Wetensch. Proc. Ser. A 73 = Indag. Math. **32** (1970), 479–495.

[Sie72a] ______, *Approximating cellular maps by homeomorphisms*, Topology **11** (1972), 271–294.

[Sie72b] ______, *Deformation of homeomorphisms on stratified sets. I, II*, Comment. Math. Helv. **47** (1972), 123–136; ibid. **47** (1972), 137–163.

[Sie80] ______, *Amorces de la chirurgie en dimension quatre: un $S^3 \times \mathbf{R}$ exotique* (*d'après Andrew H. Casson et Michael H. Freedman*), Séminaire Bourbaki (1978/79), Lecture Notes in Math., vol. 770, Springer, Berlin, 1980, pp. Exp. No. 536, pp. 183–207.

[Sma61] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406.

[Sta73] M. A. Stanko, *Approximation of compacta in E^n in codimension greater than two*, Mat. Sb. (N.S.) **90(132)** (1973), 625–636, 640.

[Tor81] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. **111** (1981), no. 3, 247–262.

[Wal64] C. T. C. Wall, *On simply-connected* 4*-manifolds*, J. London Math. Soc. **39** (1964), 141–149.

[Whi35a] J. H. C. Whitehead, *A certain open manifold whose group is unity*, Quart. J. Math. **6** (1935), 268–279.

[Whi35b] ______, *Certain theorems about three-dimensional manifolds, I*, Quart. J. Math. **5** (1935), 308–320.